Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 5
Channel representations and Norms for matrices part II

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1. **Equivalence between representations of quantum channels**
Let us first show that the Choi-Jamiołkowski map $J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{Y} \otimes \mathcal{X})$ is a linear bijection between the CPT maps on the one hand and the set of quantum states on $\mathcal{Y} \otimes \mathcal{X}$ with partial-trace over $\mathcal{Y}$ is maximally mixed on the other hand.

   a) Show that the inverse map can be defined by $\tilde{T}(X) = \mathrm{Tr}_{\mathcal{X}}[J(T)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)]$ and makes J a bijection as described above.

   Solution: We have to show that the map J is both injective and surjective under suitable restrictions. For injectivity (1), we show that $\tilde{T} = T$, for surjectivity (2) we use a result from a previous sheet and the Kraus decomposition.

   ad (1): We have that

$$\begin{aligned}\tilde{T}(X) &= \sum_{ikl} (\mathbb{1} \otimes \langle i|)(T(|k\rangle\langle l|) \otimes |k\rangle\langle l|)(\mathbb{1} \otimes X^T)(\mathbb{1} \otimes |i\rangle) \\ &= \sum_{ikl} T(|k\rangle\langle l|)\langle i|k\rangle\langle l| X^T |i\rangle \\ &= \sum_{kl} T(|k\rangle\langle l|)\langle l| X^T |k\rangle \\ &= T(\sum_{kl} X_{kl} |k\rangle\langle l|) = T(X),\end{aligned}$$

   where the last line holds by the linearity of T.

   ad (2): Let us now show that J is surjective. To this end, choose a state $\rho \in L(\mathcal{X} \otimes \mathcal{Y})$ with $\mathrm{Tr}_{\mathcal{Y}}\, \rho = \mathbb{1}/d$ with $d = \dim \mathcal{Y}$. We will show that there exists a quantum channel that has ρ as its Choi-Jamiołkowski isomorph. Express $\rho = \sum_i \lambda_i |t_i\rangle\langle t_i|$ in its eigenbasis.

   We now make use of a fact proved on problem sheet 3, namely that for an arbitrary pure quantum state $|\psi\rangle$ we find an operator Y such that $|\psi\rangle = (Y \otimes \mathbb{1})|\Omega\rangle$. In particular, we can find operators K_i such that $\sqrt{d}(K_i \otimes \mathbb{1})|\Omega\rangle = \sqrt{\lambda_i}|t_i\rangle$. (Recall that this is just the inverse of the vectorisation map $\mathrm{vec} : L(\mathcal{X}) \cong \mathcal{X} \otimes \mathcal{X}^* \to \mathcal{X} \otimes \mathcal{X}$ that acts on a basis as $|i\rangle\langle j| \mapsto |i\rangle|j\rangle$.)

   Due to the partial trace condition on ρ, the K_is fulfil

$$\sum_i K_i K_i^\dagger = d\, \mathrm{Tr}_{\mathcal{Y}} (K_i \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| (K_i^\dagger \otimes \mathbb{1}) \tag{1}$$

$$= \mathrm{Tr}_{\mathcal{Y}} \sum_i \lambda_i |t_i\rangle\langle t_i| \tag{2}$$

$$= \mathrm{Tr}_{\mathcal{Y}}\, \rho = \mathbb{1}/d, \tag{3}$$

   where for the first step we inserted a funky looking identity $\mathbb{1} = d\,\mathrm{Tr}_{\mathcal{Y}} |\Omega\rangle\langle\Omega|$ between the Ks.

   So the set $\{K_i\}$ satisfies the condition that we require of Kraus operators and, thus, define a CPT channel.

Let $\rho_T \in \mathcal{Y} \otimes \mathcal{X}$ be the Choi-Jamiołkowski state corresponding to the quantum channel T.

b) Determine a set of Kraus operators representing T.

Solution: Decompose $\rho_T = \sum_i \lambda_i |t_i\rangle\langle t_i|$ and let $K_i = \sqrt{\lambda_i}\text{vec}^{-1}(|t_i\rangle)$, where vec^{-1} denotes the inverse map of vectorisation.

c) Determine a unitary U_T representing T via the Stinespring representation.

Solution: The isometry $V : L(\mathcal{X}) \to L(\mathcal{Y} \otimes \mathcal{Z})$, as in the definition of the Stinespring representation, is given by $V = \sum_i K_i \otimes |i\rangle$, where $|i\rangle$ are orthonormal vectors in $\mathcal{Z}$ as is easily checked.

We now construct the unitary from the Stinespring representation as $U : L(\mathcal{X} \otimes \mathcal{Z}') \to L(\mathcal{Y} \otimes \mathcal{Z})$ by orthogonal completion (e.g. using Gram Schmidt) such that $U(\mathbb{1} \otimes |0\rangle) = V$ with $|0\rangle \in \mathcal{Z}'$.

Now, let U_T be a unitary representing T in the Stinespring representation.

d) Determine the Choi-Jamiołkowski state representing T.

Solution: We obtain the isometry $V = U_T(\mathbb{1} \otimes |0\rangle) = \sum_i K_i \otimes |i\rangle$ and then set $\rho_T = (\text{vec}K_i)_i(\text{vec}K_i)^\dagger$.

The rank of a quantum channel is defined as the rank of its Choi matrix.

e) Show that a quantum channel with rank r can be represented as a Stinespring dilation using an auxiliary system of dimension r.

Solution: We have $\rho_T = \sum_{i=1}^r \lambda_i |t_i\rangle\langle t_i|$, and hence $K_i = \sqrt{\lambda_i}\text{vec}^{-1}(t_i), i = 1, \ldots, r$. Now define $V = \sum_{i=1}^r K_i \otimes |i\rangle$ as an isometry from $\mathcal{X}$ to $\mathcal{Y} \otimes \mathbb{C}^r$.

2. **Examples of quantum channels**

Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$\begin{aligned} F_\epsilon(A) &:= \epsilon XAX + (1-\epsilon)A \\ D_\epsilon(A) &:= \epsilon \operatorname{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A \\ A_\epsilon(A) &:= \epsilon \operatorname{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A, \end{aligned}$$

where $\epsilon \in [0, 1]$.

a) For each channel, show that it is CPT.

b) For each channel, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

Hint: It may help to consider $\epsilon = 1$ in a first step and then generalize to arbitrary $\epsilon \in [0, 1]$.

Solution: Let Id be the identity channel with $J(\text{Id}) = |\Omega\rangle\langle\Omega|$. Then the Choi states are given by the convex combination of the channel with $\epsilon = 1$ and $J(\text{Id})$, e.g. $J(F_\epsilon) = \epsilon J(F_1) + (1-\epsilon)J(\text{Id})$. Now,

$$J(F_1) = \frac{1}{d}\sum_{ij} |i(i \oplus 1)\rangle \langle j(j \oplus 1)| , \quad J(D_1) = \frac{1}{d}\mathbb{1}, \quad J(A_1) = |0\rangle\langle 0| \otimes \mathbb{1},$$

where $\oplus$ denotes the addition modulo 2 a.k.a. xor.

We have the following possible Kraus representations of the channels

$$F(B) = XBX^\dagger, \quad D(B) = XBX + YBY + ZBZ + \mathbb{1}B\mathbb{1}$$

$$A(B) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

The isometries defining the Stinespring representation are: $V_F = X \otimes |0\rangle$, $V_D = \mathbb{1} \otimes |0\rangle + X \otimes |1\rangle + Y \otimes |2\rangle + Z \otimes |3\rangle$, likewise for V_A.

The corresponding unitaries are given, for example by $U_F = X \otimes |0\rangle\langle 0|$,

$U_D =$ SWAP if the environment is prepared in the maximally mixed state.

c) Give a physical interpretation and a good name for each channel.

Solution: Bit-flip channel, depolarizing channel, amplitude damping channel.

3. **Schatten p-norms**

On the last excercise sheet we have studied the ℓ_p-norms on vector spaces. The ℓ_p-norms have important cousins on matrix spaces, the Schatten p-norms. As they are important distant measures in quantum information, we study there different definitions and properties in this excerice.

One way to introduce the Schatten p-norm with $p \in [1, \infty)$ for a matrix $A \in \mathbb{C}^{n\times n}$ is

$$||A||_p := \left(\mathrm{Tr}\left[|A|^p\right]\right)^{\frac{1}{p}}, \tag{4}$$

where $|A| := \sqrt{A^\dagger A}$ is the matrix absolute value. Furthermore, the case $p = \infty$ is defined as the limit $||A||_\infty = \lim_{p\to\infty} ||A||_p$.

These norms are related to the ℓ_p-norms of the eigenvalues (or more generally the singular values) of A.

a) Let A be a Hermitian matrix and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector of its eigenvalues. Show that

$$||A||_p = ||\lambda||_{\ell_p} \tag{5}$$

for all p.

Solution: Let $A = U\Lambda U^\dagger$ be the eigenvalue decomposition of A. Recall that for Hermitian A it holds that $\sqrt{A^\dagger A} = U \,\mathrm{diag}(|\lambda_1|, \ldots, |\lambda_n|) U^\dagger$ as can be easily checked by squaring the equation. Then,

$$||A||_p^p = \mathrm{Tr}\left[(A^\dagger A)^{\frac{p}{2}}\right] = \mathrm{Tr}\left[U\Lambda^p U^\dagger\right] = \mathrm{Tr}\,\Lambda^p = \sum_i \lambda_i^p = ||\lambda||_{\ell_p}^p. \tag{6}$$

With this characterisation we have also established that the Schatten p-norms are invariant under unitary transformations.

b) Give the statement and proof for the Hölder inequality for Schatten p-norms.

Hint: Actually, prooving the Hölder inequality rigorously involves prooving the "von Neumann-inequality", which turns out to be quite intricate. In this exercise you can simply use it:
Let A and B be two matrices and let $s(A)$ and $s(B)$ be the vector of singular values of A and B, respectively, ordered decreasingly. Then it holds that

$$|\,\mathrm{Tr}\,[AB]\,| \leq \mathrm{Tr}\,|AB| \leq \sum_i s_i(A)s_i(B). \tag{7}$$

For a proof (sketch) of this inequality, you can take a look at Bhatia's book on matrix analysis. Slightly more direct proofs using doubly stochastic matrices were worked out by Mirsky. A more elementary proof was given R. D. Grigorieff in a note in '92. You can find it on his webpage.

Solution: There are different matrix version of Hölder's inequality: Let $1 \geq p \geq \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, then
Matrix Hölder I:

$$||A^\dagger B||_1 \leq ||A||_p ||B||_q. \tag{8}$$

Matrix Hölder II:

$$\left|\mathrm{Tr}\left[A^\dagger B\right]\right| \leq ||A||_p ||B||_q. \tag{9}$$

Just as a side remark, there is even the more general version that holds for every unitarily invariant norm $||\cdot||$.
Matrix Hölder III:

$$||A^\dagger B|| \leq ||(A^\dagger)^p||^{\frac{1}{p}} ||B^q||^{\frac{1}{q}}. \tag{10}$$

Proof of matrix Hölder I & II.
With the help of the von Neumann inequality, it is easy to reduce matrix Hölder to the standard Hölder inequality for vectors:

$$\mathrm{Tr}\,|AB| \leq |\langle s(A)|s(B)\rangle| \leq ||s(A)||_{\ell_p} ||s(B)||_{\ell_q} = ||A||_p ||B||_q. \tag{11}$$

The second version follows from the first version by showing that

$$\left|\mathrm{Tr}\left[A^\dagger B\right]\right| \leq \mathrm{Tr}\left|A^\dagger B\right|. \tag{12}$$

The most important Schatten p-norms have other interesting expressions:

c) Show that the Schatten 2-norm or Frobenius norm fulfils

$$||A||_2^2 = \sum_{i,j=1}^{n} |A_{ij}|^2. \tag{13}$$

Solution:

$$||A||_2^2 = \mathrm{Tr}\, A^\dagger A = \sum_{i,j} \bar{A}_{ji} A_{ji} = \sum_{i,j} |A_{ij}|^2. \tag{14}$$

In general, one can define the operator norms induced by the ℓ_p-norms:

$$||A||_{\ell_p \to \ell_q} = \sup_{||x||_{\ell_p}=1} ||Ax||_{\ell_q}. \tag{15}$$

d) What is the Schatten p-norm equal to $||\cdot||_{\ell_2 \to \ell_2}$?

Solution:

$$||A||_{\ell_2 \to \ell_2} = \sup_{||x||_{\ell_2}=1} ||Ax||_{\ell_2} = \sup_{||x||_{\ell_2}=1} \sqrt{\langle x, A^\dagger A\, x\rangle} \tag{16}$$

$$= \sqrt{\lambda_{\max}(A^\dagger A)} = |\lambda_{\max}(A)|. \tag{17}$$

Another important properties of Schatten p-norms is *sub-multiplicativity*, $||AB||_p \leq ||A||_p ||B||_p$ for all p and $A, B \in \mathbb{C}^{n\times n}$. Sometimes the term *matrix norm* is exclusively used for sub-multiplicative norms on matrix spaces.

e) Show the sub-multiplicativity of the Schatten p-norms.

Solution: Using the min-max principle of the Rayleigh quotient (also called Courant-Fischer theorem), we first establish that $|\lambda_i(AB)| \leq ||A||_\infty |\lambda_i(B)|$. Proof:

$$|\lambda_i(AB)| = \min_{U, \dim U = k} \max_{x \in U, ||x||_{\ell_2}=1} ||ABx||_2 \tag{18}$$

$$\leq \min_{U, \dim U = k} \max_{x \in U, ||x||_{\ell_2}=1} ||A||_\infty ||Bx||_2 \tag{19}$$

$$= ||A||_\infty |\lambda_i(B)|, \tag{20}$$

where we have used that $|\langle x, Ax\rangle| \leq ||A||_\infty |\langle x, x\rangle|$, which follows from the operator norm definition of the spectral norm.

Now we have

$$||AB||_p = \left[\sum_i |\lambda_i(AB)|^p\right]^{\frac{1}{p}} \leq ||A||_\infty \left[\sum_i |\lambda_i(B)|^p\right]^{\frac{1}{p}} \tag{21}$$

$$= ||A||_\infty ||B||_p \leq ||A||_p ||B||_p. \tag{22}$$

In the last step we have used the ordering of the p-norms inherited by the ordering of the ℓ_p-norms, in particular $||A||_\infty \leq ||A||_p$ for all $p < \infty$.