Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 6
Entanglement witnesses

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1. **An entanglement witness for the maximally entangled state**
   Let $|\Omega\rangle$ be the maximally entangled state.
   a) Show that $W = \mathbb{1} - d\,|\Omega\rangle\langle\Omega|$ is a witness for $|\Omega\rangle$.

   Solution: We first check that the expectation value of W is positive for seperable states. To this end let $\rho = (|\psi\rangle \otimes |\phi\rangle)(\langle\psi| \otimes \langle\phi|)$ be a pure product state. Then,

$$\langle W\rangle_\rho = (\langle\psi| \otimes \langle\phi|)(\mathbb{1} - d\,|\Omega\rangle\langle\Omega|)(|\psi\rangle \otimes |\phi\rangle) \tag{1}$$
$$= 1 - |\langle\psi|\phi\rangle|^2 \geq 0, \tag{2}$$

   since by Cauchy-Schwarz's inequality $|\langle\psi|\phi\rangle| \leq ||\psi||_{\ell_2}||\phi||_{\ell_2} \leq 1$.
   Second,

$$\langle W\rangle_{|\Omega\rangle} = 1 - d < 0. \tag{3}$$

   b) Give an example of an entangled state that is not detected by W.

   Solution: Let's try a orthogonally twisted copy of $|\Omega\rangle$ in the two qubits setting. We have $\langle\Omega|\psi^+\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle) = 0$ and, thus, $\langle W\rangle_{\psi^+} = 1 \geq 0$.

2. **The reduction map as a witness**
   The *reduction map* is defined as $\Lambda_R(X) = \mathrm{Tr}(X)\mathbb{1} - X$.
   a) Show that Λ_R is positive but not completely positive, in other words it is a witness.

   Solution: Regarding the positivity, we calculate for a positive semi-definite X and an arbitrary vector $|\psi\rangle$: $\langle\psi|\Lambda(X)|\psi\rangle = \mathrm{Tr}(X)||\psi||^2 - \langle\psi|X|\psi\rangle$. Since X is positive semi-definite, $\mathrm{Tr}(X) = ||X||_1 \geq ||X||$ and, hence, the first term is lower-bounded by $||\psi||||X||$. At the same time the second term is upper bounded by $|\langle\psi|X|\psi\rangle| \leq ||\psi||||X\psi|| \leq ||\psi||^2||X||$. We conclude that Λ is a positive map.
   One straigt-forward way to check for complete positivity is by Choi's theorem. To this end, we calculate

$$J(\Lambda_R) = \Lambda_R \otimes \mathbb{1}(|\Omega\rangle\langle\Omega|) \tag{4}$$
$$= \sum_{i,j} \frac{1}{d}\mathrm{Tr}(|i\rangle\langle j|)\mathbb{1} \otimes |i\rangle\langle j| - |\Omega\rangle\langle\Omega| \tag{5}$$
$$= \sum_{i} \frac{1}{d}\mathbb{1} \otimes |i\rangle\langle i| - |\Omega\rangle\langle\Omega| \tag{6}$$
$$= \frac{1}{d}\mathbb{1}_{d^2} - |\Omega\rangle\langle\Omega|. \tag{7}$$

   Note that $dJ(\Lambda_R)$ is W from the previous excercise. It's expectation value in state $|\Omega\rangle$ is not positive. Thus, the map is not completely positive.
   b) Give at least one example for states that are detectable by Λ_R.

Solution: We have already seen that $|\Omega\rangle$ is an example. But we should think maybe also of less obvious examples.

c) Give at least one example for entangled states that are not detected by Λ_R.

Solution: Consider $\rho = \mathbb{1} + \alpha\mathbb{F}$ for $\alpha \in [-1,1]\setminus\{0\}$ up to normalization.

d) What is the observable witness associated to the positive map by the Choi-Jamiolkowski-isomorphism?

Solution: Note that $(A^*, \mathbb{1}) = \mathrm{Tr}(A) = \overline{(\mathbb{1}, A)}$ for A. With this in mind, we calculate $(\Lambda_R(A), B) = (\mathrm{Tr}(A)\mathbb{1} - A, B) = \overline{\mathrm{Tr}(A)}\,\mathrm{Tr}(B) - (A, B) = (A, \mathbb{1}\,\mathrm{Tr}(B) - B) = (A, \Lambda_R(B))$. Thus, $J(\Lambda_R^*) = J(\Lambda_R) = W$ from excericse 2.

A map Λ is called *decomposable* if it can be written as $\Lambda = P_1 + P_2 \circ T$, where P_1, P_2 are completely positive maps and T is the transpose.

e) Show that any state that is detected by Λ_R can also be detected by the partial transpose criterion.
Hint: Argue first that Λ_R is decomposable.

Solution: Since $\mathrm{Tr}\,X = \mathrm{Tr}\,X^T$, we have the decomposition $\Lambda_R = \tilde{\Lambda} \circ T$ with $\tilde{\Lambda}(X) = \mathrm{Tr}(X)\mathbb{1} - X^T$. The Choi matrix of $\tilde{\Lambda}$ is $J(\tilde{\Lambda}) = \mathbb{1} - \mathbb{F}$ with $\mathbb{F}$ the flip operator. This can be seen by a short calculation. Using $\mathbb{F}^2 = \mathbb{1}$, we now observe that $(\mathbb{1} - \mathbb{F})^2 = 2(\mathbb{1} - \mathbb{F})$. Correspondingly, every eigenvalue λ of $J(\tilde{\Lambda})$ fulfils $\lambda = \lambda^2/2 \geq 0$. Thus, $\tilde{\Lambda}$ is completely positive. This establishes that Λ_R is decomposable into the composition of a completely positive map and the transposition.

In general, we have the following statement: Let Λ be a decomposable map and ρ an entangled state detected by Λ, i.e. $\Lambda \otimes \mathbb{1}(\rho) = P_1 \otimes \mathbb{1}(\rho) + (P_2 \otimes \mathbb{1})(T \otimes \mathbb{1})(\rho))$ is not positive semi-definite. Then, also $T \otimes \mathbb{1}(\rho)$ is not positive semi-definite. Thus, ρ is also detected by the partial transpose criterion. In other words, any decomposable witness gives rise to a weaker criterion than the partial transpose criterion.

This applies to Λ_R.

f) Translate the condition of a map Λ being decomposable to a criterion for the observable witness $J(\Lambda^*)$. What is the implication of a self-adjoint observable witness being decomposable in this sense?

Solution: Applying the linear Choi-Jamiołkowski isomorphism yields $J(\Lambda) = J(P_1) + J(P_2 \circ T) = J(P_1) + (\mathbb{1} \otimes T)J(P_2)$. Thus, we call a bipartite operator Wdecomposable, if it can be written as $W = A_1 + A_2^{T_B}$, where A_1 and A_2 are positive semi-definite and T_B denotes the partial transposition on the second system. We have the criterion that any state detected by a decomposable W has negative partial transpose.

3. **Constructing entanglement witnesses from the partial transpose**
In the entanglement theory of bi-partite systems the partial transpose criterion plays a prominet rôle. Let $T : L(\mathcal{H}) \to L(\mathcal{H})$ be the transposition map $X \mapsto T(X) = X^T$. The partial transpose is the map $T : L(\mathcal{H} \otimes \mathcal{H}) \to L(\mathcal{H} \otimes \mathcal{H})$. Let $(\cdot, \cdot)$ be the Hilbert-Schmidt inner-product on $L(\mathcal{H})$ defined as $(X, Y) = \mathrm{Tr}(X^T Y)$. The adjoint Λ^* of a map

$\Lambda : L(\mathcal{H}) \to L(\mathcal{H})$ with respect to $(\cdot,\cdot)$ is defined such that $(\Lambda(X), Y) = (X, \Lambda^*(Y))$ holds for all $X, Y \in L(\mathcal{H})$.

In the lecture, we saw that for any positive but not completely positive map Λ, $J(\Lambda^*)$ is also witness. But $J(\Lambda^*)$ is not necessarily detecting all the states that Λ detects. Let Λ detect ρ_e and let $|\eta\rangle$ be an eigenvector with negative eigenvalue of $(\mathbb{1} \otimes \Lambda)(\rho_e)$. $|\Omega\rangle$ shall be the maximally entangled state.

a) Show that $\mathbb{1} \otimes T$ is self-adjoint, i.e. $(\mathbb{1} \otimes T)^* = \mathbb{1} \otimes T$.

Solution: We start by convincing ourself that

$$(A^T, B) = \mathrm{Tr}(\overline{A}B) = \sum_{i,j} \langle i|\overline{A}|j\rangle \langle j|B|i\rangle \tag{8}$$

$$= \sum_{i,j} \langle j|A^\dagger|i\rangle \langle i|B^T|j\rangle = \mathrm{Tr}(A^\dagger B^T) \tag{9}$$

$$= (A, B^T). \tag{10}$$

(One could also be convinced that this is true by observing that the trace is invariant under transposition.) By linearity it suffice to consider tensor product matrices $A \otimes B$ and $C \otimes D$ and show that

$$(\mathbb{1} \otimes T(A \otimes B), C \otimes D) = (A, C) \cdot (B^T, D) = (A, C) \cdot (B, D^T) \tag{11}$$

$$= (A \otimes B, \mathbb{1} \otimes T(C \otimes D)). \tag{12}$$

b) Show that $J(\Lambda^*)$ detects $\hat{\rho}_e = (\mathbb{1} \otimes X^\dagger)\rho_e(\mathbb{1} \otimes X)$ where X is defined such that $|\eta\rangle = \mathbb{1} \otimes X |\Omega\rangle$.

Solution: By assumption

$$\begin{aligned}
0 &> \mathrm{Tr}[(\Lambda \otimes \mathbb{1})(\rho_e) |\eta\rangle\langle\eta|] \\
&= \mathrm{Tr}[\rho_e(\Lambda^* \otimes \mathbb{1})((\mathbb{1} \otimes X) |\Omega\rangle\langle\Omega| (\mathbb{1} \otimes X^\dagger))] \\
&= \mathrm{Tr}[\rho_e(\mathbb{1} \otimes X)(\Lambda^* \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|)(\mathbb{1} \otimes X^\dagger)] \\
&= \mathrm{Tr}[(\mathbb{1} \otimes X^\dagger)\rho_e(\mathbb{1} \otimes X)(\Lambda^* \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|)]
\end{aligned}$$

With this we have established the missing direction in the proof of the lectures' theorem relating positive maps and observables as witnesses.

It is also possible to construct an observable that detects ρ_e itself from Λ. To this end, we define $\mathcal{W}_e = (\mathbb{1} \otimes \Lambda^*)(|\eta\rangle\langle\eta|)$.

c) Show that this construction, in fact, gives rise to an entanglement witness $\mathcal{W}_e$ for ρ_e.

Solution:

$$0 > \mathrm{Tr}[(\Lambda \otimes \mathbb{1})(\rho_e) |\eta\rangle\langle\eta|] = \mathrm{Tr}[\rho_e(\Lambda^* \otimes \mathbb{1})(|\eta\rangle\langle\eta|)]$$

As an application of this construction we consider the following setting. In our (fictitious) lab, we are trying to prepare a two-qubit state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We use a simple model[1] for what is actually happening in the lab, namely that we prepare a state with some noise

$$\rho(p) \coloneqq p\,|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}.$$

Our goal is to have an observable witness that decides whether $\rho(p)$ is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled PPT states. Therefore, if $\rho(p)$ is entangled, the partial transpose $\mathbb{1} \otimes T$ will always detect $\rho(p)$..

d) Assume $|\psi\rangle$ has Schmidt decomposition $|\psi\rangle = a\,|01\rangle + b\,|10\rangle$. Determine the values of p depending on a, b such that $\rho(p)$ is entangled.

Solution: We know from the lecture that $\rho(p)$ is entangled iff $\rho(p)^{T_B} \ngeq 0$.

$$\begin{aligned}\rho(p)^{T_B} =& p(|a|^2\,|01\rangle\langle 01| + |b|^2\,|10\rangle\langle 10| + a\bar{b}\,|00\rangle\,\langle 11| + b\bar{a}\,|11\rangle\,\langle 00|) + \frac{1-p}{4}\sum_{ij}|ij\rangle\langle ij| \\ =& (p|a|^2 + (1-p)/4)\,|01\rangle\langle 01| + (p|b|^2 + (1-p)/4)\,|10\rangle\langle 10| \\ &+ (1-p)/4\,|00\rangle\langle 00| + (1-p)/4\,|11\rangle\langle 11| + pa\bar{b}\,|00\rangle\,\langle 11| + p\bar{a}b\,|11\rangle\,\langle 00|\end{aligned}$$

The eigenvalues are given by

$$(1-p)/4, (1-p)/4 \pm p|a||b|$$

with eigenvector $|\eta\rangle = 1/\sqrt{2}\,|00\rangle - |11\rangle$ corresponding to the potentially negative eigenvalue. Hence, $\rho(p)$ is entangled iff $p > 1/(1+4|a||b|)$.

e) Use the eigenvector corresponding to a negative eigenvalue of $(\mathbb{1} \otimes T)(\rho(p))$ in order to derive an entanglement witness $\mathcal{W}$ for $\rho(p)$.

Solution: $\mathcal{W} = |\eta\rangle\langle\eta|^{T_B} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\,\langle 10| + |10\rangle\,\langle 01|) = \frac{1}{2}(\mathbb{1} - |01+10\rangle\langle 01+10|)$

f) Show that, in fact, the witness $\mathcal{W}$ detects *all* entangled states of the form $\rho(p)$.

Solution:

$\mathrm{Tr}[|\eta\rangle\langle\eta|^{T_B}\,\rho(p)] = \mathrm{Tr}[|\eta\rangle\langle\eta|\,\rho_p^{T_B}] < 0$ iff $\rho(p)$ is entangled .

[1]What is the corresponding noise channel for this model?