Freie Universität Berlin

Tutorials on Quantum Information Theory

Winter term 2020/21

Problem Sheet 3
Schmidt Decomposition, Teleportation, and Introduction to Graphical Calculus

J. Eisert, A. Nietner, C. Bertoni, F. Arzani, R. Suzuki

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1. **Schmidt decomposition** (9 Points: 2+1+1+2+1+2)

   In the lecture, you already saw the Schmidt decomposition of bipartite quantum states $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as given by

   $$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\psi_j^1\right\rangle \left|\psi_j^2\right\rangle,$$

   where $\{\left|\psi_j^i\right\rangle\}$ are orthonormal bases of $\mathcal{H}_i$.

   In this exercise, we will study some useful properties and applications of the Schmidt decomposition. First, some warm up

   a) Find a Schmidt decomposition of the following two qubits states:

   $$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \\ |\psi_2\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \end{aligned} \tag{1}$$

   Let us now look at states with the same Schmidt coefficients, that is

   $$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\psi_j^1\right\rangle \left|\psi_j^2\right\rangle, \quad |\Phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\phi_j^1\right\rangle \left|\phi_j^2\right\rangle.$$

   b) Show that $|\Psi\rangle$ and $|\Phi\rangle$ are related by a local unitary, i.e., a unitary of the form $U \otimes V$ with U and V unitary. Give that unitary explicitly.

   c) Show that any local unitary transformation leaves the Schmidt coefficients invariant.

   d) Determine the reduced density matrices $\rho_1 = \mathrm{Tr}_2 |\Psi\rangle\langle\Psi|$ and $\rho_2 = \mathrm{Tr}_1 |\Psi\rangle\langle\Psi|$. How can the Schmidt coefficients be interpreted? What are the Schmidt coefficients of the maximally entangled state?

   e) Use the Schmidt decomposition to show that *any* bipartite state $|\Psi\rangle$ can be expressed as

   $$|\Psi\rangle = (A \otimes \mathbb{1}) |\Omega\rangle,$$

   for a matrix A, where $|\Omega\rangle$ is a maximally entangled state.

   f) Use the Schmidt decomposition to show that a pure bipartite state $|\psi\rangle_{AB}$ is a product state if and only if the reduced states $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ are pure.

2. **General teleportation schemes** (7 Points: 1+2+1+1+2)

   In the lecture you saw a teleportation scheme using a maximally entangled state shared by Alice and Bob. In this exercise we will generalise this setting to teleportation schemes with higher local dimensions.

We begin by reformulating the qubit teleportation scheme in terms of Bell-basis measurements. The Bell basis for two qubits is given by

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

a) Show that the Bell basis can be prepared starting from $|\Phi_0\rangle$ using local Pauli operations only.

In the lecture, you saw the scheme in which Alice applies $(H \otimes \mathbb{1}^{\otimes 2})(CX \otimes \mathbb{1})$ to $|\psi\rangle |\Phi_0\rangle$ and then measures in the Z-basis. She then communicates her results, say a, b on the two registers to Bob, who applies $X^a Z^b$ as a correction to obtain $|\psi\rangle$ on his side.

The two schemes are equivalent via the identification of outcomes

$$00 \leftrightarrow 0, 10 \leftrightarrow 1, 01 \leftrightarrow 2, 11 \leftrightarrow 3,$$

where we used (a).

This reformulation generalises to a d-dimensional teleportation scheme in which Alice and Bob share a maximally entangled state $|\omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. As above the scheme is based on measuring in a maximally entangled orthonormal basis set $\{|\Psi_\alpha\rangle\}_{\alpha=1}^{d^2}$, i.e., an orthonormal basis for which $\mathrm{Tr}_1[|\Psi_\alpha\rangle\langle\Psi_\alpha|] = \mathbb{1}_d = \mathrm{Tr}_2[|\Psi_\alpha\rangle\langle\Psi_\alpha|]$.

There exist several constructions of linearly independent sets $\{U^\alpha\}_{\alpha=1}^{d^2}$ of d^2 trace-wise orthogonal unitary operators $U^\alpha \in U(d)$,

$$\mathrm{Tr}[U^{\alpha\dagger} U^\beta] = \mathrm{Tr}[U^{\beta\dagger} U^\alpha] = \delta_{\alpha\beta} \mathbb{1}$$

for all α and β. In the following, we just assume the existence of such a set.

b) Show that such a set $\{U^\alpha\}_{\alpha=1}^{d^2}$ gives rise to a maximally entangled basis set by setting

$$|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1} |\omega\rangle .$$

The maximally entangled state $|w\rangle$ has the following properties, which are important for quantum teleportation scheme.

c) Show that for an arbitrary unitary $U \in U(d)$, $(U \otimes \mathbb{1}) |\omega\rangle = (\mathbb{1} \otimes U^T) |\omega\rangle$.

d) Show that for an arbitrary pure state of Alice $|\phi\rangle_A$, $(\langle\phi|_A \otimes \mathbb{1}_B) |\omega\rangle = \frac{1}{\sqrt{d}} |\phi^*\rangle_B$ and $\langle\omega| (|\phi\rangle_A \otimes \mathbb{1}_B) = \frac{1}{\sqrt{d}} \langle\phi^*|_B$, where $|\phi^*\rangle$ is the complex conjugate of $|\phi\rangle$

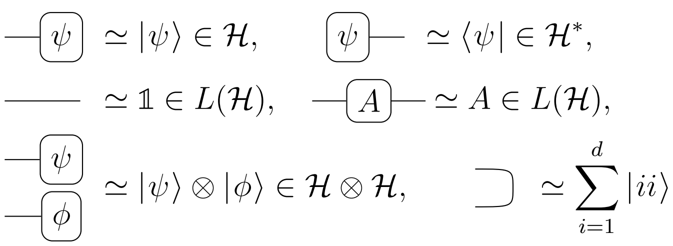
Now consider the setting in which Alice and Bob share the state $|\omega\rangle_{AB}$ and Alice measures her part of the system in the basis $\{|\Psi_\alpha\rangle\}_{AA'} =$ to send her state $|\psi\rangle_{A'}$ to Bob.

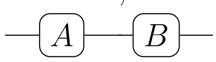
e) Insert the resolution of the identity $\sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|_{AA'}$ and use the result from (c) and (d) to show that $|\psi\rangle_{A'} |\omega\rangle_{AB} = \frac{1}{d} \sum_\alpha |\Psi_\alpha\rangle_{AA'} \otimes (U^\alpha)^*_B |\psi\rangle_B$. Then, describe how to perfom d-dimensional quantum teleportation.

3. **Introduction to graphical calculus with tensor networks** (6 Points: 1+1+1+2+1)

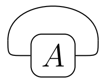
As you might have noticed, already for a little number of tensor factors even simple calculations can become hard to follow quite easily. Hence, an alternative approach to visualize such calculations was developed. Namely, graphical calculus with tensor networks, often atributed to Roger Penrose. We will give a short introduction into the basics of this calculation technique in this exercise. However, we encourage you to have a look into `https://arxiv.org/pdf/1603.03039.pdf`, which gives a nice and complete overview over tensor networks. For this course, you won't need most of the content but it constitutes a good reference where you can find any concept we will use (in particular, in chapter 1 and 2).

In tensor network notation, a tensor is simply an object that has indices, usually a set of complex numbers $A_{i_1,\dots,i_n}$. A tensor with one index is a vector, one with two indices is a matrix. A tensor with n indices is denoted as a box with n legs, hence we have the following correspondences

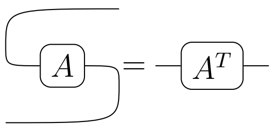


One can think of each unconnected leg carrying a (dual) Hilbert space. Connecting two legs denotes contraction of the indices, so that for example the matrix product $(AB)_{ij} = \sum_k A_{ik}B_{kj}$ is denoted by 

a) Draw the expectation value $\langle\psi|\, A \,|\psi\rangle$ as a tensor network.

b) What does the following tensor network represent? 

c) Prove



d) Using tensor networks, prove the following statement from Exercise sheet 1

$$\mathrm{Tr}(\rho_{AB} O_A \otimes \mathbb{1}_B) = \mathrm{Tr}(\mathrm{Tr}_B(\rho_{AB}) O_A) \tag{2}$$

Hint: recall that any ρ_{AB} *can be decomposed in bases of* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *as* $\rho_{AB} = \sum_{ijkl} \rho_{ijkl} \,|i\rangle\!\langle j| \otimes |k\rangle\!\langle l|$.

e) Prove that $\mathrm{Tr}(A^2) = \mathrm{Tr}((A \otimes A)F)$ using tensor networks. F denotes the *flip operator* exchanging the two subsystems, i.e. $F : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$.