Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 4
Kraus representation and Norms for matrices part I

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1. **On the Kraus representation of quantum channels**(10 points : 1 + 2 + 2 + 2 + 1 + 1 + 1)
Recall that a map $\mathcal{C} : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$ is a proper quantum channel if and only if it is completely positive and trace preserving, which is equivalent to

$$\mathcal{C} : \rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{1}$$

for some *Kraus operators* E_k such that $\sum_k E_k^\dagger E_k = \mathbb{I}$. In the following, we investigate the operational meaning of Kraus operators. For simplicity, we restrict ourselves to quantum channels with the same input and output space $L(\mathcal{X})$. Suppose we apply a unitary U to the joint system and environment in the state $\rho \otimes |0\rangle\langle 0| \in L(\mathcal{X} \otimes \mathcal{Z})$, where $|0\rangle \in \mathcal{Z}$ is some reference state, and then we measure system $\mathcal{Z}$ in the computational basis.

   a) Show that the action of any unitary on the joint system can be written as

   $$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle \langle l| \ ,$$

   with respect to the basis $\{|i\rangle\}_i$ on the second system for a set of operators $\{E_k\}$.

   b) Now, we perform a von-Neumann (that is, projective) measurement on $\mathcal{Z}$ in the same basis. Determine the post-measurement state conditioned on outcome i.

   c) What is the probability of obtaining outcome i? What does this entail for the operators E_k?

   d) Give the corresponding operational interpretation of the Kraus operators E_k and the unitary U.

   e) Now, suppose we want to implement a projective measurement on $\mathcal{X}$ via a global unitary and a projective measurement on $\mathcal{Z}$. Consider the unitaries $U \in U(\mathcal{X} \otimes \mathcal{Z})$ on the joint system that give rise to this situation. What conditions do they have to satisfy?

   Hint: the measurement needs to collapse the state of the first system as well.

   f) Can you think of an example for the case of $\mathcal{X}$ and $\mathcal{Z}$ being each a qubit?

   We will show one last property of the Kraus representation

   g) Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{i=1}^N$ be two sets of linear operators in $L(\mathcal{X}, \mathcal{Z})$ fulfilling the completeness relation of Kraus operators. Show that if the two sets are related by a unitary transformations $U \in U(N)$ such that $\tilde{K}_i = \sum_j U_{ij} K_j$, the channels represented by the sets coincide.

2. **From ℓ_p to Schatten norms, to trace distance** (10 points: 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1)
In quantum information we deal with a handful of different matrix spaces such as the set of quantum states and also quantum channels. For quantitative statements we

have to equip these spaces with distance measures. Depending on the application and context different distance measures have the desired operational meaning.

A prominent role is played by the so called *Schatten p-norms.* But to set the stage we first introduce their analogue on vector spaces, namely *ℓ_p-norms.* For $1 \leq p < \infty$ the ℓ_p-norm on the complex vector space $\mathbb{C}^n$ is defined as

$$||\bullet||_{\ell_p} : x \mapsto ||x||_{\ell_p} := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}},$$

and the ℓ_∞-norm as

$$||\bullet||_{\ell_\infty} : x \mapsto ||x||_{\ell_\infty} := \lim_{p\to\infty} ||x||_{\ell_p}.$$

a) Show that $||\bullet||_{\ell_\infty} = \max_{1\leq i\leq n} |x_i|$.

It is not hard (although tedious) to show that $||\bullet||_{\ell_p}$ satisfies all properties of a norm (it is positive definite, absolutely homogeneous, subadditive aka triangle inequality). Schatten p-norms are defined for linear operators acting on a (finite-dimensional) vector space $\mathcal{V}$ in a similar manner, namely

$$\|\bullet\|_p : L(\mathcal{V}) \to [0, \infty)$$
$$O \mapsto \|O\|_p = (\mathrm{Tr}[|O|^p])^{\frac{1}{p}}$$

where $|O| = \sqrt{O^\dagger O}$.

b) Show that $\|O\|_p = ||\boldsymbol{\sigma}_O||_{\ell_p}$ where $\boldsymbol{\sigma}_O = (\sigma_O(1), \ldots, \sigma_O(n))$ are the singular values of O.

Hint: start by writing the singular value decomposition $O = U\Sigma V$ in Dirac (bra-ket) notation, then write $O^\dagger O$ and apply the definition of p norm.

A notable special case is $p = 1$, that is $\|O\|_1 = \mathrm{Tr}[|O|]$c, which turns out to give a useful measure of distinguishability between quantum states, called *trace distance* $\|\rho - \sigma\|_1$. The remaining exercises of this sheet will focus on this.

c) Show that $0 \leq \|\rho - \sigma\|_1 \leq 2$ for any pair of density matrices.

In the following, we will prove that the normalized trace distance provides an achievable upper bound for the probability of obtaining the same outcome if *any* measurement (POVM) is performed on ρ vs σ. Suppose Alice flips a coin and, depending on the result, sends either ρ or σ to Bob. Bob wants to perform a measurement that will tell him which one of the two states he has. To this end, he implements a POVM with two operators, M_0 and M_1, such that the outcome 0 means the state is ρ and the outcome 1 means the state is σ.

d) Show that the probability that Bob successfully determines which state he has is

$$P_{\text{success}} = \frac{1}{2}(1 + \mathrm{Tr}[M_0(\rho - \sigma)]) \tag{2}$$

The trace distance measures the optimal probability of distinguishing the states, in equations this reads

$$\frac{1}{2}\|\rho - \sigma\|_1 = \max_{0\leq M\leq \mathbb{1}} \mathrm{Tr}[M(\rho - \sigma)]. \tag{3}$$

We will prove this in a few steps.

e) Write $|\rho - \sigma|$ in terms of the the positive and negative parts of $\rho - \sigma$, P and Q.

f) Show that $\mathrm{Tr}[P] = \mathrm{Tr}[Q]$.

g) Consider the projector on the support of P, Π_P. Use the previous three points to show that $\mathrm{Tr}[\Pi_P(\rho - \sigma)] = \frac{1}{2}\|\rho - \sigma\|_1$.

Hint: try to write each side in terms of $\mathrm{Tr}[P]$.

We are almost done: Π_P achieves our upper bound. We just need to show that Π_P is also the optimal POVM element:

h) Show that any positive operator M such that $M \leq \mathbb{I}$ will obey

$$\mathrm{Tr}[M(\rho - \sigma)] \leq \frac{1}{2}\|\rho - \sigma\|_1 = \mathrm{Tr}[\Pi_P(\rho - \sigma)]. \tag{4}$$

Hint: use again $\rho - \sigma = P - Q$ *and inequalities for the trace of positive operators we've seen in a previous sheet.*

As a concluding remark, to make the connection with the first exercise we note that this statement can be turned into statements for the distinguishability of quantum *channels*.

Finally, we can give an operational interpretation to orthogonal states: the following points are to show that Bob can perfectly distinguish ρ and σ if and only if they are orthogonal, i.e. $\mathrm{Tr}(\rho\sigma) = 0$.

i) Show that if $\mathrm{Tr}(\rho\sigma) = 0$, then $||\rho - \sigma||_1 = 2$. *Hint: first, show that* $\mathrm{Tr}(\rho\sigma) = 0 \implies \rho\sigma = 0$

j) Conversely, show that $||\rho - \sigma||_1 = 2$ implies $\mathrm{Tr}(\rho\sigma) = 0$. What does this imply for the probability of distinguishing ρ and σ?

Hint: recall that $||\rho - \sigma||_1 = 2\,\mathrm{Tr}(\Pi_P(\rho - \sigma))$, *use this to show that* $\mathrm{Tr}(\Pi_P\rho) = 1$ *and* $\mathrm{Tr}(\Pi_P\sigma) = 0$.