Freie Universität Berlin

Tutorials on Quantum Information Theory

Winter term 2021/22

Problem Sheet 5

Channel representations and Norms for matrices part II

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1. **Equivalence between representations of quantum channels** (11 Points: 1+1+2+1+2+2+1+1)
The aim of this exercise is to establish a duality between quantum channels and quantum states. To this end, let

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i, i\rangle \tag{1}$$

be the maximally mixed state on a bipartite system $\mathcal{H}_d \otimes \mathcal{H}_d$ and denote by $\Omega = |\Omega\rangle\langle\Omega|$ its corresponding density matrix. Then define the Choi-Jamiołkowski map as

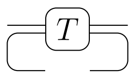
$$J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{X} \otimes \mathcal{Y}) :: T \mapsto (T \otimes \mathbb{1})(\Omega) \tag{2}$$

with Ω now the maximally entangled state in $\mathcal{X} \otimes \mathcal{X}$ and $\mathbb{1}$ the identity on $X \otimes X^*$. Throughout let d be the dimension of $\mathcal{X}$.

We will show that J as a map from the completely positive tracepreserving (CPTP) maps to the set of quantum states on a bipartite system $\mathcal{X} \otimes \mathcal{Y}$ with the restriction $\mathrm{Tr}_{\mathcal{Y}}\, \rho = \mathbb{1}/d$ is a bijection.

a) Use the criterion for positivity from the lecture and show that for a CPTP map T from operators on $\mathcal{X}$ to operators on $\mathcal{Y}$, $J(T) \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Y})$ is indeed a density matrix on the joined system.

b) Use the diagrammatic notation to first draw the action of T on a density matrix $\rho \in L(\mathcal{X})$. Then use that intuition to draw the Choi-state $J(T)$ in diagrammatic notation. (Hint: you can represent T diagramatically as

$T =$  (3)

where the two bottom legs can be thought as corresponding to the "input" space $L(\mathcal{X}) \simeq \mathcal{X} \otimes \mathcal{X}^*$ and similarly for the two top legs. It may be convenient to think about how the density matrix Ω is expressed graphically.)

c) Show that J is injective. (Hint: Do so by showing that for any $J(T)$ in the image of J you can define a $\tilde{T}$ that maps $X \in L(\mathcal{X})$ to $\tilde{T}(X) = d\,\mathrm{Tr}_{\mathcal{X}}\left[J(T)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)\right]$. If you use this hint, explain what this implies?).

d) Before we show surjectivity of J we want to get used to some concepts from the lecture: determine a set of Kraus operators representing T (Hint: use the matrix representation of pure states on a bipartite system from two weeks ago together with the eigendecomposition of ρ_T.).

e) Assuming $\dim(\mathcal{X}) = \dim(\mathcal{Y})$, show that J is surjective. (Hint: Assume a given ρ with the restriction mentioned above and use the previous exercise to construct a CPTP map T such that $J(T) = \rho$.).

Let $\rho_T \in \mathcal{Y} \otimes \mathcal{X}$ be the Choi-Jamiołkowski state corresponding to the quantum channel T.

f) Determine a unitary U_T representing T via the Stinespring representation.

Now, let U_T be a unitary representing T in the Stinespring representation.

g) Determine the Choi-Jamiołkowski state representing T from U_T.

The rank of a quantum channel is defined as the rank of its Choi matrix.

h) Show that a quantum channel with rank r can be represented as a Stinespring dilation using an auxiliary system of dimension r.

2. **Examples of quantum channels** (9 Points: 1+4+2+2)
Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$
$$D_\epsilon(A) := \epsilon \operatorname{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$
$$A_\epsilon(A) := \epsilon \operatorname{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A,$$

where $\epsilon \in [0, 1]$.

a) For each channel, show that it is CPT.

Next, we represent each quantum channel in different three ways, discussed in the previous exercise.

b) For each channel with fixed $\epsilon = 1$, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

c) Generalise the previous results to an arbitrary $\epsilon \in [0, 1]$. (Hint: First compute the respective representations for $\epsilon = 0$ and then reason for the Choi state on the one hand, and for the Kraus and Stinespring representations on the other hand how to combine the $\epsilon = 0$ and $= 1$ cases into an arbitrary ϵ case.)

d) For arbitrary $\epsilon \in [0, 1]$ compute the action of each channel on the inputs $|0\rangle\langle 0|$ and $\rho = \mathbb{1}/2$. What is the physical interpretation of each channel?