Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 7
Partial Transpose, Entanglement Witness, and Majorisation

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1. **Warm-up: Partial transpose** (2 Points)
In the entanglement theory of bi-partite systems the partial transpose criterion plays a prominet role. Let $T : L(\mathcal{H}) \to L(\mathcal{H})$ be the transposition map $X = \sum_{i,j} c_{ij} |i\rangle\langle j| \mapsto X^T = \sum_{i,j} c_{ij} |j\rangle\langle i|$. The partial transpose T_A is defined as the following:

$$T_A : L(\mathcal{H}_A \otimes \mathcal{H}_B) \to L(\mathcal{H}_A \otimes \mathcal{H}_B)$$
$$Y = \sum_{i,j} c_{ijkl} |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B \mapsto Y^{T_A} = \sum_{i,j} c_{ijkl} |j\rangle\langle i|_A \otimes |k\rangle\langle l|_B$$

The partial transpose T_B is defined likewise, i.e., the transposition map on H_B.

   a) For a pure state $\rho_1 = \frac{1}{4}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)_A \otimes (|0\rangle + |1\rangle)(\langle 0| + \langle 1|)_B$ and a maximally entangled state $\rho_2 = \frac{1}{2}(|00\rangle_{AB} + |11\rangle_{AB})(\langle 00|_{AB} + \langle 11|_{AB})$, find the partial transpose T^A of them. Then, calculate eigenvalues of each transposed matrix, and compare two cases.

2. **Constructing entanglement witness from the partial transpose** (10 Points: 1+2+2+2+2+1)
In the lecture, we saw that every separable bi-partite quantum state has a positive partial transpose, which means that the positivity is an entanglement criterion. First, we show that this criterion is valid.

   a) Show that for an arbitrary separable bi-partite quantum state $\rho = \sum_i p_j(\rho_{Ai} \otimes \rho_{Bi})$, all eigenvalues of ρ^{T_A} are greater than or equal to 0, i.e., $\rho^{T_A} \geq 0$.

   In general, the opposite direction is not true. However, if we restrict a quantum state to a pure state, the opposite is also true as the following.

   b) Show that a bi-partite pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is separable if it has a positive partial transpose.

   Hint: Prove the contraposition: if $|\psi\rangle$ is entangled, $(|\psi\rangle\langle\psi|)^{T_A}$ has at least one negative eigenvalue. To this end, use Schmidt decomposition.

   Recall that an entanglement witness is an observable W with the following conditions: (i) $\mathrm{Tr}(W\rho) \geq 0$ for all separable states σ and (ii) there exists an entangled state ρ satisfying $\mathrm{Tr}(W\rho) < 0$.

   c) Consider an entangled state ρ. Let $|\mu\rangle$ be an eigenvector of ρ^{T_A} whose eigenvalue is negative. Then show that $W = (|\mu\rangle\langle\mu|)^{T_A}$ is an entanglement witness and $|\mu\rangle$ is an entangled state.

   As an application of this witness, we consider the following setting. In our (fictitious) lab, we are trying to prepare a two-qubit state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We use a simple model for what is actually happening in the lab, namely that we prepare a state with some noise

$$\rho(p) := p\,|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}.$$

Our goal is to have an observable witness that decides whether $\rho(p)$ is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled. positive partial transpose (PPT) states. Therefore, the partial transpose T^A will always detect entanglement of $\rho(p)$.

d) Assume $|\psi\rangle = a\,|01\rangle_{AB} + b\,|10\rangle_{AB}$. Calculate eigenvalues of $\rho(p)^{T_B}$, and determine the values of p depending on a, b such that $\rho(p)$ is entangled.

   Hint: Use the fact that $\rho(p)$ is entangled if and only if $\rho(p)^{T_B} \ngeq 0$.

e) Use the eigenvector corresponding to a negative eigenvalue of $(\rho(p))^{T_B}$ in order to derive an entanglement witness $\mathcal{W}$ for $\rho(p)$.

f) Show that, in fact, the witness $\mathcal{W}$ detects *all* entangled states of the form $\rho(p)$.

3. **Majorisation and transforming quantum states by local unitaries.** (8 Points: 2+2+2+2)

In this problem we will look at the task of transforming a given copy of a pure bipartite quantum state $|\psi\rangle$ to another quantum state $|\phi\rangle$ using LOCC. The question is: Under which conditions is the transition $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ possible?

The key to the answer of this question is the concept of majorisation. We say that a vector $x \in R^n$ majorises $y \in R^n$ (denoted by $x \succ y$) if for all $k = 1, \ldots, n$, $\sum_{j=1}^{k} x_j^{\downarrow} \geq \sum_{j=1}^{k} y_j^{\downarrow}$ and $\sum_{j=1}^{n} x_j^{\downarrow} = \sum_{j=1}^{n} y_j^{\downarrow}$. Here, $x^{\downarrow}$ denotes the sorted version of x, i.e., a permutation of the elements of x such that $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$. From now on, let x and y be non-negative vectors.

a) Show that $x = (\frac{2}{3}, \frac{1}{3}, 0)^T$ majorises $y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$.

One can show that $x \prec y$ if and only if $x = \sum_j p_j \Pi_j y$ for a probability distribution p and permutation matrices Π_j (Please accept this equation.). By Birkhoff's theorem, which lies at the heart of majorisation theory, that statement is equivalent to saying that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix D[1].

For two Hermitian operators $X, Y \in L(C^d)$ we say that $X \prec Y$ if $\lambda(X) \prec \lambda(Y)$, where $\lambda(A)$ is the spectrum of a matrix A.

b) Show that $X \prec Y$ if and only if there exists a probability distribution p and unitary matrices U_j such that

$$X = \sum_j p_j U_j Y U_j^{\dagger}.$$

   Hint: For "only if" direction, do eigenvalue decomposition as $X = U\Lambda_X U^{\dagger}$ and $Y = V\Lambda_Y V^{\dagger}$, and use the fact "$\lambda(X) \prec \lambda(Y)$ if and only if $\lambda(X) = \sum_j p_j \Pi_j \lambda(Y)$". For "if" direction, use again eigenvalue decomposition and the fact "$\lambda(X) \prec \lambda(Y)$ if and only if $\lambda(X) = D\lambda(Y)$ for some doubly stochastic matrix D".

We are now ready to prove the (surprising!) theorem: $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ if and only if $\mathrm{Tr}_B[|\psi\rangle\langle\psi|] \prec \mathrm{Tr}_B[|\phi\rangle\langle\phi|]$. (We encourage you to have a look into `https://arxiv.org/pdf/quant-ph/9811053.pdf`, which is the original paper of the thorem.)

c) Show the "only if" direction using the previous result. You can suppose that LOCC is realised by a measurement on Alice's side and a corresponding unitary

[1] A matrix D is called doubly stochastic if $\forall i, j D_{ij} \geq 0$ and $\forall i \sum_j D_{ij} = \sum_j D_{ji} = 1$, i.e., all rows and columns are probability distributions.

on Bob's side. In other words, from Alice's point of view it must be the case that [2]

$$M_j \mathrm{Tr}_B[|\psi\rangle\langle\psi|]M_j^\dagger = p_j \mathrm{Tr}_B[|\phi\rangle\langle\phi|].$$

Hint: Use the polar decomposition of $M_j\sqrt{\mathrm{Tr}_B[|\psi\rangle\langle\psi|]}$.

d) Now show the "if" direction by proceeding analogously.

[2]This is because the transition from $|\psi\rangle$ to $|\phi\rangle$ comes about as a post-measurement state with probability p_j.