Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 10
Aspects of quantum algorithms and circuits

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1. **More stabilizers** (15 Points: 3+4+1+4+2+1)

Because the stabilizer formalism introduced in the previous sheet is so important in quantum computation (and quantum error correction, as you will see later in the course), the present problem is again devoted to this . We will start by proving an important property of stabilizer sets, then compute stabilizers for some familiar states and finally introduce a very powerful vector notation to work with Pauli strings and Clifford transformations.

a) Let $\mathcal{S} \subset \mathcal{P}_n$ be a subgroup of strings from the Pauli group such that $\boldsymbol{s}_i \boldsymbol{s}_j = \boldsymbol{s}_j \boldsymbol{s}_i$ for all $\boldsymbol{s}_i, \boldsymbol{s}_j \in \mathcal{S}$ which is generated by d independent elements[1]. Show that the subspace stabilized by $\mathcal{S}$ has dimension $\dim = 2^{n-d}$. (*Hint:* Think about how to write the projector onto the stabilized subspace in terms of a generating set and use that dim equals the trace of the projector)

b) Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{1}$$

be the standard Bell state, $|\mathrm{GHZ}_n\rangle$ be the n qubit GHZ state

$$|\mathrm{GHZ}_n\rangle = \frac{|00\ldots0\rangle + |11\ldots1\rangle}{\sqrt{2}} \tag{2}$$

and let $|\phi\rangle$ be the 3 qubit parity state

$$\langle x_1, x_2, x_3|\phi\rangle = \begin{cases} \frac{1}{\sqrt{2}^3}\,, & \text{if } x_1 + x_2 + x_3 \; = 0 \mod 2 \\ 0\,, & \text{else.} \end{cases} \tag{3}$$

Write down a set of stabilizers for $|\psi\rangle$, $|\mathrm{GHZ}_n\rangle$ and $|\phi\rangle$. Next find a set of stabilizers for the state

$$|\chi\rangle = \frac{|1\rangle \otimes |\mathrm{GHZ}_{n-2}\rangle \otimes |0\rangle + |0\rangle \otimes |\mathrm{GHZ}_{n-2}\rangle \otimes |1\rangle}{\sqrt{2}} \tag{4}$$

Note that every Pauli string, and hence every stabilizer, can be represented (up to a phase) by a so called *check vector*. This is a $2n$ dimensional binary vector $\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{v})$ constructed as follows. Start with the all-zero vector. If the stabilizer has an X at the i'th site change the i'th entry to a 1. Likewise if the stabilizer has a Z at the i'th site change the $n+i$'th entry to a 1. If the i'th postition of the stabilizer is a Y, change both i'th and the $n+i$'th entries to a 1. The overall phase can be encoded in an additional degree of freedom. In summary, a general n-qubit Pauli string $\boldsymbol{s}$ can be written

$$\boldsymbol{s} \; = e^{i\phi}\,(X_1^{u_1} Z_1^{v_1} \otimes \cdots \otimes X_n^{u_n} Z_n^{v_n}) \tag{5}$$

for some $\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{v})$ and $\phi \in [0, 2\pi)$.

[1] Remember that the generators $\gamma_1, \ldots, \gamma_d$ of a group $\mathcal{G}$ are group elements such that any $g \in \mathcal{G}$ can be obtained as a product of some γ_js and their inverses. *Independent* generators are such that none of them can be obtained as a product of the others, even up to a phase (you can always obtain an independent set by removing dependent elements).

c) Write down the check vectors for the stabilizers of a generating set of stabilizers associated to $|\mathrm{GHZ}_n\rangle$.

d) Compute HXH and HZH for H the Hadamard matrix. Let $S_i = X_i X_{i+1}$ for $i = 1, \dots n-1$ and let $S_n = \prod_i Z_i$. Use the first part of this exercise in order to compute

$$H^{\otimes n} S_i H^{\otimes n} \tag{6}$$

for all i and write down the corresponding check vectors.

Since the Clifford group maps Pauli strings to Pauli strings, we can represent the transformation $\boldsymbol{s} \mapsto C\boldsymbol{s}C^\dagger$ (modulo phases) as a transformation of the check vector $\boldsymbol{x}$ corresponding to $\boldsymbol{s}$. As it turns out, such transformation is linear and therefore represented by a $2n \times 2n$ binary matrix M. In other words, we have $\boldsymbol{s} \mapsto C\boldsymbol{s}C^\dagger \Rightarrow \boldsymbol{x} \mapsto M\boldsymbol{x}$.

e) Show that the map $\boldsymbol{s} \mapsto C\boldsymbol{s}C^\dagger$ is indeed linear. Write down the matrix M corresponding to the action of $H^{\otimes n}$ on a Pauli string (neglecting the effect of phases).

Incorporating the phases in this formalism for the general case is beyond the scope of the present sheet. In the last point below we look at a specific example.

f) Explain how the phase ϕ transform when a general Pauli string $\boldsymbol{s}$ is conjugated by $H^{\otimes n}$ as $\boldsymbol{s} \mapsto H\boldsymbol{s}H$. (*Hint*: In addition to the previous cases of H conjugating X and Z consider the action of H when conjugating Y.)

2. **Universal gate set** (6 points (2+1+2+1))

The aim of this exercise is to show that the gate set $\{CNOT, H, T\}$ is universal, i.e. we can approximate any gate to an arbitrary degree of accuracy just by using these three gates. The strategy is to show that we can use H and T to generate any single qubit gate, and the conclusion follows from the insight that $CNOT$ along with arbitrary one qubit gates is universal. For the sake of ease we will use a slightly modified convention for the T gate throughout the exercise. This is, we use the definition

$$T = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}. \tag{7}$$

Convince yourself that this is just the standard definition of the T gate modified by a phase factor $e^{-i\pi/8}$.

We will start by showing that any unitary U can be written as

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \tag{8}$$

where $R_z(\theta) = e^{-i\frac{\theta}{2}Z} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$, $R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$.

a) Let $U \in U(2)$ be a one qubit unitary, show that there exist real numbers x, y, z, t such that

$$U = \begin{pmatrix} e^{i(x-y-t)}\cos z & -e^{i(x-y+t)}\sin z \\ e^{i(x+y-t)}\sin z & e^{i(x+y+t)}\cos z \end{pmatrix} \tag{9}$$

b) Show that any one qubit unitary U can be expressed as

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \tag{10}$$

for some real numbers $\alpha, \beta, \gamma, \delta$.

It is possible, but tedious, to show that we can find an analogous decomposition using any pair of linearly independent axes $\hat{n}$ and $\hat{m}$.

We will now see how to approximate an arbitrary single-qubit rotation around two linearly independent axes by using the Hadamard gate and the T gate. A single-qubit rotation can be written as $R_{\overrightarrow{n}}(\theta) \equiv \exp(-i\theta \overrightarrow{n} \cdot \overrightarrow{\sigma}/2) = \cos(\theta/2)\mathbb{1} - i\sin(\theta/2)(n_x X + n_y Y + n_z Z)$, and any single qubit gate can be written as a rotation around some axis.

c) Calculate $THTH$, and find θ and $\overrightarrow{n} = (n_x, n_y, n_z)$ with respect to it.

Hint: Use that $T = e^{-i\pi/8Z}$ *and* $HTH = e^{-i\pi/8X}$.

Observe that the $\frac{\theta}{2\pi}$ in the previous point is an irrational number.

d) Show that you can approximate an arbitrary rotation about the axis $\overrightarrow{n}$ in the previous point by some product of the operators H and T.

Let us define another rotation about an axis $\overrightarrow{m}$ as $R_{\overrightarrow{m}}(\theta) = HR_{\overrightarrow{n}}(\theta)H$. Because H is a rotation about $X + Z$ axis, the axis $\overrightarrow{m}$ is not equal to $\overrightarrow{n}$. Then from the points (b) we can generate an arbitrary single-qubit unitary by $R_{\overrightarrow{m}}$ and $R_{\overrightarrow{n}}$.