Freie Universität Berlin

Tutorials on Quantum Information Theory

Winter term 2021/22

Problem Sheet 12

Measurement-based quantum computing

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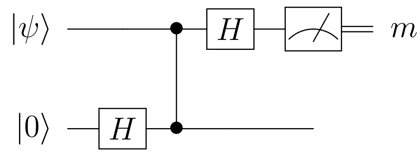
In the circuit picture of quantum computation that you have seen in the course until now, algorithms are carried out by applying unitaries to "static" qubit registers. Measurement based quantum computing (MBQC) is an alternative paradigm of quantum computation wherein unitary evolution is the by-product of measurements performed on parts of entangled states. The whole process can still be described by a circuit, a fact that we will use to derive some fundamental results in the following. However, the shift in perspective has important implications for physical implementations.

1. **Gate teleportation** (12 points: 4+2+2+1+3)

The fundamental primitive of MBQC is called gate teleportation, a simple version of which can be demonstrated in the following circuit:



(1)

Here, the entangling gate is a controlled-Z acting on a two-qubit state as $CZ\,|ab\rangle = (-1)^{ab}\,|ab\rangle$.

a) Suppose that in the above circuit we measure the first register in the Z eigenbasis. Write the resulting state on the remaining subsystem in terms of the input state, depending on the measurement outcome m (you can neglect the normalization constant)[1].

b) Imagine taking the output state of the second wire, denoted $|\psi'(m_1)\rangle$, following a measurement with outcome m_1, and feeding it back to a similar circuit, with measurement outcome m_2. Can you write the output state in terms of $|\psi\rangle$? *Hint: you shouldn't need to do any calculation.*

A key insight in MBQC is that if we want to repeat the above process n times we can prepare an entangled n-qubit resource state $|\Gamma\rangle$ *beforehand*, independent of the input state $|\psi\rangle$. $|\Gamma\rangle$ can be depicted as a one-dimensional strip of pair-wise entangled qubits, called a 1-d *cluster state*. We can then entangle $|\psi\rangle$ to the first qubit of the strip and subsequently only perform measurements (and possibly single-qubit Pauli corrections to remove the dependency of the output on measurement outcomes). Since $\langle Z = \pm 1|\, H = \langle X = \pm 1|$, you can convince yourself that in circuit 1 after the CZ the first qubit is effectively measured in the X basis. In the following point, we consider the H gates right before the computational basis measurement as "part of an X measurement process".

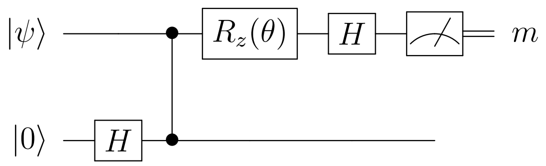
c) Draw a sketch of the circuit resulting from the n-fold repetition of the citcuit in Eq. 1 and write an expression for the resource state $|\Gamma\rangle$ (*Hint: CZ gates on different qubits all commute and isolate all measurements at the end of the circuit.*)

[1] In the context of MBQC, the measurements are often assumed to be "destructive", in the sense that the measured qubits are consumed by the measurement process and therefore not included in the description of what happens next. This reflects the physical reality where qubits might for example be encoded in travelling photons which are absorbed by a detector during the measurement.

d) Consider $R_z(\theta) = \exp(-\mathrm{i}\frac{\theta}{2}Z)$ and define $X_\theta = R_z(\theta)^\dagger X R_z(\theta)$. Show that

$$X_\theta R_z^\dagger(\theta)\, H \,|Z = m\rangle = (-1)^m R_z^\dagger(\theta)\, H \,|Z = m\rangle \,. \tag{2}$$

e) Consider the following circuit



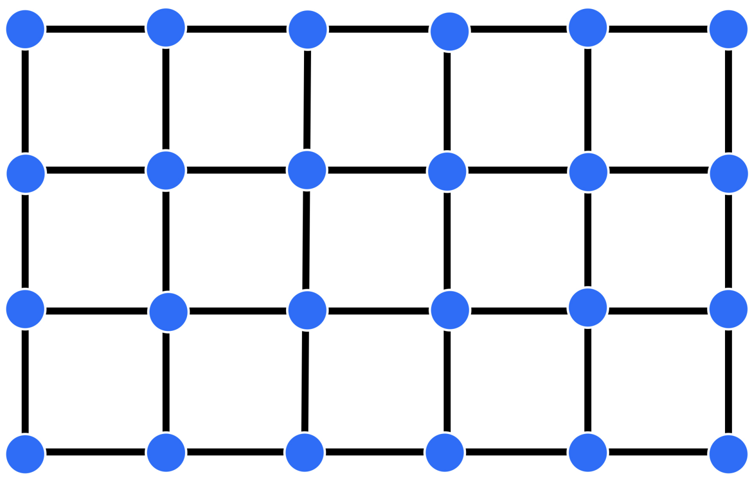
(3)

where the measurement is in the computational basis. What is the observable that is effectively measured on the first qubit after the CZ? And what is the output state of the circuit, depending on the measurement outcome? *Hint: the $R_z(\theta)$ commutes with CZ.*

Since we get the Hadamard gate "for free" according to circuit 1, this result, together with the results in previous sheets shows us that a 1-d cluster state and single-qubit measurements are sufficient to perform an arbitrary single-qubit operation. The result is furthermore deterministic if we can operate corrective X operations depending on measurement outcomes.

2. **Universal Quantum Computation with Cluster States** 8 Points: 3 + 1 + 2 + 2

In this exercise we consider a two dimensional cluster state where qubits are arranged in a rectangular grid



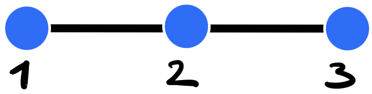
where nodes are qubit registers. This state can be obtained by preparing each qubit in the $|+\rangle$ state and applying CZ operations between qubits connected by an edge. This general procedure can be used to produce states represented by any graph, which are simply called *graph states*. Cluster states, described by a rectangular grid, lie at the core of measurement based quantum computation (MBQC) because, given one such state, one can perform *any* quantum computation with single qubit measurements (in various bases), provided the rectangular patch is large enough. In the present exercise, we will sketch a proof of this fact. According to the previous exercise, it is sufficient to show that we can perform two-qubit gates. To this end, let us start with some definitions.

The stabilizer formalism is once again useful to compactly describe what is going on. The stabilizer generators of an arbitrary graph state $|\Gamma\rangle$, with Γ some graph, are given by

$$\mathcal{S} = \{X_a \prod_{i\sim a} Z_i \mid a \in \Gamma\}, \tag{4}$$

where $i \sim a$ denotes the set of qubits adjacent (connected) to qubit a. In paticular, it holds that $S_a |\Gamma\rangle = |\Gamma\rangle \ \forall S_a \in \mathcal{S}$.

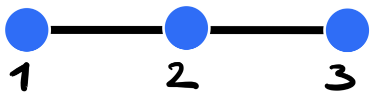
a) Consider the graph state represented by



Write down the stabilizer generators of this state according to Eq. 10 and check that they indeed stabilize the state. *Hint: write the stabilizer of the state by conjugating the ones of the* $|+++\rangle$ *state by the appropriate* CZ*s.*

We now use stabilizers to prove two useful tricks that allow us to easily modify the shape of a 2-d cluster state. Remember that when we have a stabilizer state with stabilizer generators $\{S_j\}$ and we want to measure some Pauli operator O, we can represent the action of a measurement as follows: if O commutes with all stabilizers, the measurement result is predetermined and the state is unchanged (being already an eigenstate of O. If O does not commute with some stabilizers, we can find a set of generators such that only one generator, say S_1, anti-commutes with O. This set can be found by multiplying some of the generators together. Following the measurement, we replace S_1 with $(-1)^m O$, where m is the measurement outcome.

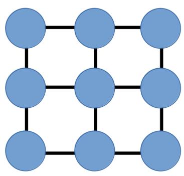
b) Consider again the graph state represented by



Show that measuring the second qubit in the X basis the stabilizers of the post-measurement state on the first and third qubit are those of a two-qubit cluster, apart for a Hadamard on either the first or the third qubit and measurement-dependent phases. *Hint: start by finding a set of generators such that only one anti-commutes with the measurement. Multiply then the post-measurement stabilizers to remove unwanted dependencies on operators acting on the measured qubit.*

The above result shows that we can "shorten" wires to connect initially distant qubits on the lattice. The second equivalence is obtained multiplying $Z_1X_2Z_3$ by $\pm X_2$.

c) Consider now the 3×3 square cluster state

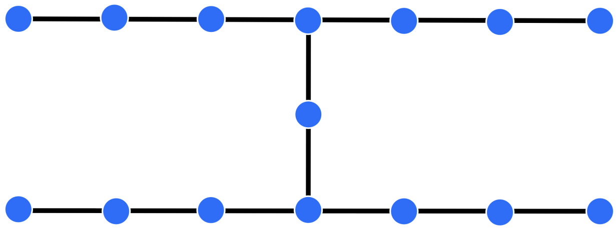


Show that measuring Z on the central node effectively disentangles it from the rest of the state, leaving the other qubits in a graph state.

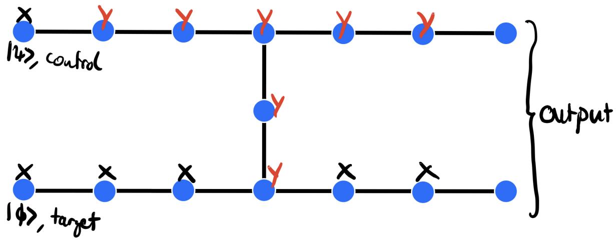
These two tricks can be generalized to show that, given a 2D cluster state, one can "cut out" any 2D regular grid and obtain the graph state needed to implement some circuit by single qubit measurements on appropriate sites. This justifies using the graph shape in the following point.

Finally, we turn to the $CNOT$ gate. We can apply the $CNOT$ gate in the MBQC scheme by using the following graph state.

d) Consider the following graph state:



Show that the following measurements implement a CNOT gate between the twoinput states $|\psi\rangle$ and $|\phi\rangle$ up to local pauli corrections:



Hint: There are two ways to prove this. Either, one explicitly calculates the output of the full circuit corresponding to the preparation and the measurements or one uses the stabilizer formalism where one only has to keep track how the stabilizers of the graph state change during the measurements. You might also have a look at https://arxiv.org/pdf/quant-ph/0301052.pdf.