Freie Universität Berlin

Tutorials on Quantum Information Theory

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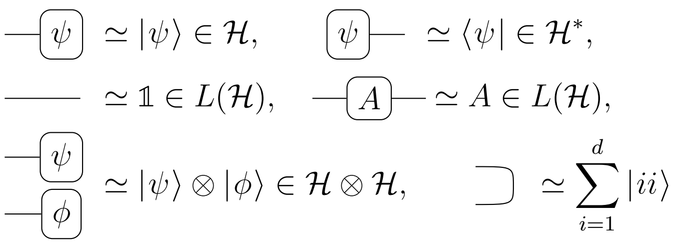
Problem Sheet 4
Graphical calculus and Quantum Channels

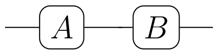
J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

1. **Introduction to graphical calculus with tensor networks** (6 Points: 1+1+1+2+1)

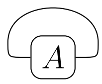
As you might have noticed, already for a little number of tensor factors even simple calculations can become hard to follow quite easily. Hence, an alternative approach to visualize such calculations was developed. Namely, graphical calculus with tensor networks, often atributed to Roger Penrose. We will give a short introduction into the basics of this calculation technique in this exercise. However, we encourage you to have a look into `https://arxiv.org/pdf/1603.03039.pdf`, which gives a nice and complete overview over tensor networks. For this course, you won't need most of the content but it constitutes a good reference where you can find any concept we will use (in particular, in chapter 1 and 2).

In tensor network notation, a tensor is simply an object that has indices, usually a set of complex numbers $A_{i_1,\ldots,i_n}$. A tensor with one index is a vector, one with two indices is a matrix. A tensor with n indices is denoted as a box with n legs, hence we have the following correspondences

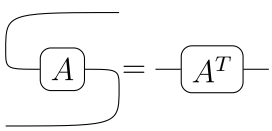


One can think of each unconnected leg carrying a (dual) Hilbert space. Connecting two legs denotes contraction of the indices, so that for example the matrix product $(AB)_{ij} = \sum_k A_{ik}B_{kj}$ is denoted by 

a) Draw the expectation value $\langle\psi| A |\psi\rangle$ as a tensor network.

b) What does the following tensor network represent? 

c) Prove



d) Using tensor networks, prove the following statement from Exercise sheet 1

$$\mathrm{Tr}(\rho_{AB} O_A \otimes \mathbb{1}_B) = \mathrm{Tr}(\mathrm{Tr}_B(\rho_{AB}) O_A) \tag{1}$$

Hint: recall that any ρ_{AB} *can be decomposed in bases of* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *as* $\rho_{AB} = \sum_{ijkl} \rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|$.

e) Prove that $\mathrm{Tr}(A^2) = \mathrm{Tr}((A \otimes A)F)$ using tensor networks. F denotes the *flip operator* exchanging the two subsystems, i.e. $F : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$.

2. **On the Kraus Representation of Quantum Channels**
We have seen in the lecture as well as in previous exercise sheets that many of the notions in quantum information theory can be understood by starting with pure-state quantum mechanics and demanding a description for subsystems of such quantum systems. Some examples of this are the following statements

- Given an arbitrary pure state $|\psi\rangle \in H_A \otimes H_B$ describing the joint state of two physical systems A and B, all measurement statistics of measurements on subsystem A (or B) are fully contained in the reduced density matrices $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ (or $\rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$). I.e. density matrices are required to describe the possible states of subsystems of larger systems whose states are pure.
- Given an arbitrary mixed state $\rho \in D(H_A)$ there always exists a second Hilbert space H_B and a pure state $|\psi\rangle \in H_A \otimes H_B$ such that $\rho = tr_B |\psi\rangle\langle\psi|$(Such a $|\psi\rangle$ is called a purification of ρ). This means that all density matrices can be interpreted as states of a subsystem of a larger system which is in a pure state.
- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system.

In this exercise we want to develop a similar picture for quantum channels by exploring the fact that quantum channels are exactly set of operations one can implement on a quantum system H_A by implementing a unitary operation on a joint system $H_A \otimes H_B$ and then looking at how the state of the subsystem A has transformed.

Recall that a map $\mathcal{C} : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$ is a proper quantum channel if and only if it is completely positive and trace preserving, which is equivalent to

$$\mathcal{C} : \rho \mapsto \sum_k E_k \rho E_k^\dagger \tag{2}$$

for some *Kraus operators* E_k such that $\sum_k E_k^\dagger E_k = \mathbb{I}$. In the following, we investigate the operational meaning of Kraus operators. For simplicity, we restrict ourselves to quantum channels with the same input and output space $L(\mathcal{X})$. Suppose we apply a unitary U to the joint system and environment in the state $\rho \otimes |0\rangle\langle 0| \in L(\mathcal{X} \otimes \mathcal{Z})$, where $|0\rangle \in \mathcal{Z}$ is some reference state, and then we measure system $\mathcal{Z}$ in the computational basis.

a) Show that the action of any unitary on the joint system can be written as

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle \langle l| \ ,$$

with respect to the basis $\{|i\rangle\}_i$ on the second system for a set of operators $\{E_k\}$. In particular show how these operators are related to the unitary U.

b) Now, we perform a von-Neumann (that is, projective) measurement on $\mathcal{Z}$ in the same basis. Determine the post-measurement state conditioned on outcome i.

c) What is the probability of obtaining outcome i? What does this entail for the operators E_k?

d) Give the corresponding operational interpretation of the Kraus operators E_k and the unitary U.

e) Now, suppose we want to implement a projective measurement on $\mathcal{X}$ via a global unitary and a projective measurement on $\mathcal{Z}$. Consider the unitaries $U \in U(\mathcal{X} \otimes \mathcal{Z})$ on the joint system that give rise to this situation. What conditions do they have to satisfy?

Hint: the measurement needs to collapse the state of the first system as well.

f) Can you think of an example for the case of $\mathcal{X}$ and $\mathcal{Z}$ being each a qubit?

We will show one last property of the Kraus representation

g) Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{i=1}^N$ be two sets of linear operators in $L(\mathcal{X}, \mathcal{Z})$ fulfilling the completeness relation of Kraus operators. Show that if the two sets are related by a unitary transformations $U \in U(N)$ such that $\tilde{K}_i = \sum_j U_{ij} K_j$, the channels represented by the sets coincide.

3. **Equivalence between representations of quantum channels** (11 Points: 1+1+2+1+2+2+1+1)
The aim of this exercise is to establish a duality between quantum channels and quantum states. To this end, let

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i, i\rangle \tag{3}$$

be the maximally mixed state on a bipartite system $\mathcal{H}_d \otimes \mathcal{H}_d$ and denote by $\Omega = |\Omega\rangle\langle\Omega|$ its corresponding density matrix. Then define the Choi-Jamiołkowski map as

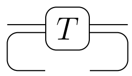
$$J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{X} \otimes \mathcal{Y}) :: T \mapsto (T \otimes \mathbb{1})(\Omega) \tag{4}$$

with Ω now the maximally entangled state in $\mathcal{X} \otimes \mathcal{X}$ and $\mathbb{1}$ the identity on $X \otimes X^*$. Throughout let d be the dimension of $\mathcal{X}$.

We will show that J as a map from the completely positive tracepreserving (CPTP) maps to the set of quantum states on a bipartite system $\mathcal{X} \otimes \mathcal{Y}$ with the restriction $\mathrm{Tr}_{\mathcal{Y}}\, \rho = \mathbb{1}/d$ is a bijection.

a) Use the criterion for positivity from the lecture and show that for a CPTP map T from operators on $\mathcal{X}$ to operators on $\mathcal{Y}$, $J(T) \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Y})$ is indeed a density matrix on the joined system.

b) Use the diagrammatic notation to first draw the action of T on a density matrix $\rho \in L(\mathcal{X})$. Then use that intuition to draw the Choi-state $J(T)$ in diagrammatic notation. (Hint: you can represent T diagramatically as

$T =$  (5)

where the two bottom legs can be thought as corresponding to the "input" space $L(\mathcal{X}) \simeq \mathcal{X} \otimes \mathcal{X}^*$ and similarly for the two top legs. It may be convenient to think about how the density matrix Ω is expressed graphically.)

c) Show that J is injective. (Hint: Do so by showing that for any $J(T)$ in the image of J you can define a $\tilde{T}$ that maps $X \in L(\mathcal{X})$ to $\tilde{T}(X) = d\,\mathrm{Tr}_{\mathcal{X}}\left[J(T)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)\right]$. If you use this hint, explain what this implies?).

d) Before we show surjectivity of J we want to get used to some concepts from the lecture: determine a set of Kraus operators representing T (Hint: use the matrix representation of pure states on a bipartite system from two weeks ago together with the eigendecomposition of ρ_T.).

e) Assuming $\dim(\mathcal{X}) = \dim(\mathcal{Y})$, show that J is surjective. (Hint: Assume a given ρ with the restriction mentioned above and use the previous exercise to construct a CPTP map T such that $J(T) = \rho$.).

Let $\rho_T \in \mathcal{Y} \otimes \mathcal{X}$ be the Choi-Jamiołkowski state corresponding to the quantum channel T.

f) Determine a unitary U_T representing T via the Stinespring representation.

Now, let U_T be a unitary representing T in the Stinespring representation.

g) Determine the Choi-Jamiołkowski state representing T from U_T.

The rank of a quantum channel is defined as the rank of its Choi matrix.

h) Show that a quantum channel with rank r can be represented as a Stinespring dilation using an auxiliary system of dimension r.