Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 5
More Quantum Channels and Entropy

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1. **Examples of quantum channels** (9 Points: 1+4+2+2)
Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$
$$D_\epsilon(A) := \epsilon \operatorname{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$
$$A_\epsilon(A) := \epsilon \operatorname{Tr}[A]\, |0\rangle\langle 0| + (1-\epsilon)A,$$

where $\epsilon \in [0,1]$.

a) For each channel, show that it is CPT.

Next, we represent each quantum channel in different three ways, discussed in the previous exercise.

b) For each channel with fixed $\epsilon = 1$, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

c) Generalise the previous results to an arbitrary $\epsilon \in [0,1]$. (Hint: First compute the respective representations for $\epsilon = 0$ and then reason for the Choi state on the one hand, and for the Kraus and Stinespring representations on the other hand how to combine the $\epsilon = 0$ and $= 1$ cases into an arbitrary ϵ case.)

d) For arbitrary $\epsilon \in [0,1]$ compute the action of each channel on the inputs $|0\rangle\langle 0|$ and $\rho = \mathbb{1}/2$. What is the physical interpretation of each channel?

2. **On Shannon entropy...**
To begin with let us first show some simple properties of entropies, in particular, of the mutual information.

Recall the definition of the Shannon entropies for random variables X, Y which take values in $\mathcal{X}, \mathcal{Y}$ and are distributed according to probability distributions p, q over $\mathcal{X}$ and $\mathcal{Y}$, respectively.

$$H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x) \text{ (Shannon entropy)} \tag{1}$$
$$H(X|Y) = H(X,Y) - H(Y) = \sum_{x\in\mathcal{X}} p(x) H(Y|X=x) \text{ (Conditional entropy)} \tag{2}$$

a) Show that $0 \leq H(X) \leq \log|\mathcal{X}|$, where the first equality holds *iff* there is an $x \in \mathcal{X}$ for which $p(x) = 1$ and the second inequality holds *iff* $p(x) = 1/|\mathcal{X}|$ for all x.

b) Show that the Shannon entropy is *subadditive*, i.e., that $H(X,Y) \leq H(X) + H(Y)$ with equality if X and Y are independent..

Hint: Show that $H(X,Y) - H(X) - H(Y) \leq 0$ *using that* $\log_2 x \ln 2 = \ln x \leq x - 1$.

c) Show that $H(Y|X) \geq 0$ equality if and only if Y is a (deterministic) function of X.

Hint: Use Bayes' rule: $p(x,y) = p(y|x)p(x)$

3. **... and the von-Neumann entropy**

For any state $\rho \in \mathcal{D}(\mathcal{H})$ with $\dim\mathcal{H} = d$ the von-Neumann entropy is defined as $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$. Throughout this problem, if the global state being referred to is clear, we will denote entropies of the reduced states using the corresponding Hilbert space as an argument, e.g. the entropy of a state ρ_{AB} reduced on subsystem A is denoted $S(A)$.

a) Show that $0 \leq S(\rho)$ with equality if and only if ρ is pure. (One can also show the upper bound $S(\rho) \leq \log d$.)

b) Show that the von-Neumann entropy is *subadditive* in the sense that if two distinct systems A and B have a joint quantum state ρ^{AB} then $S(A, B) \leq S(A) + S(B)$, with equality if $\rho_{AB} = \rho_A \otimes \rho_B$.

Hint: You may use the inequality $S(\rho) \leq -\operatorname{Tr}[\rho \log \sigma]$ for an arbitrary quantum state σ and that for two matrices A and B, $\log(A \otimes B) = \log(A) \otimes \mathbb{1} + \mathbb{1} \otimes \log(B)$.

c) Suppose that $p = (p_i)_i$ is a probability vector and the states ρ_i are mutually orthogonal. Show that

$$S\left(\sum_i p_i \rho_i\right) = H(p) + \sum_i p_i S(\rho_i).$$

and use this result to infer that

$$S\left(\sum_i p_i \rho_i \otimes |i\rangle\langle i|\right) = H(p) + \sum_i p_i S(\rho_i),$$

where $\langle i|j\rangle = \delta_{ij}$ and the ρ_i are arbitrary quantum states.

d) Use the results from b) and c) to infer that the von-Neumann entropy S is concave, that is, $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$ for a probability distribution $\{p_i\}$.

e) let Ω_{AB} be the maximally entangled state on two Hilbert spaces of equal dimension d, i.e. $\Omega = |\Omega\rangle\langle\Omega|$ with

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle . \tag{3}$$

Compute $S(A|B)$. What do you conclude?

f) Let ρ_{AB} be a bipartite state. Use the result of point c) to show that if ρ_A is separable, i.e. $\rho_{AB} = \sum_i p_i \sigma_A^i \otimes \tau_B^i$ where σ^i and τ^i are states, then $S(A|B) \geq 0$.