

Problem Sheet 6
Operator Properties and LOCC

J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

1. Operator Properties

In this exercise we take a short break from following the main content covered in the lecture and return back to proving some simple but useful identities for operators on complex Hilbert spaces. In particular, we explore the two important facts that operators are completely specified by their diagonal elements in all bases as well as the power of the square root representation for positive operators.

- a) Interestingly in a complex inner product space an operator is fully specified when its diagonal elements in all bases are known. Show this by verifying the identity

$$\langle \psi | A | \phi \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \langle \psi + i^k \phi | A | \psi + i^k \phi \rangle. \quad (1)$$

- b) Use the previous identity to show that $\forall \psi : \langle \psi | A | \psi \rangle = \langle \psi | B | \psi \rangle \implies A = B$.
- c) Use this to show that the class of operators $A \in L(\mathcal{H})$ which preserve the inner product is exactly the set of unitaries. I.e. if $\forall \psi, \phi : \langle A\psi | A\phi \rangle = \langle \psi | \phi \rangle$ then A is unitary and vice versa.
- d) A useful property of positive operators is the following: If A is a positive operator then there exists a unique positive operator $A^{1/2}$ which satisfies $A^{1/2}A^{1/2} = A$. Moreover this operator satisfies $[A, H] = 0 \implies [A^{1/2}, H] = 0$. Use this to show that the product of two positive operators is positive if and only if they commute. (hint: Also show that $A \geq B \wedge B \geq A \implies A = B$).
- e) Even though the product of two positive operators is not necessarily positive, the following holds $A \geq 0 \wedge B \geq 0 \implies \text{Tr} AB \geq 0$. Show this.

2. Local operations and classical communication (LOCC).

At the heart of entanglement theory lies the notion of LOCC. To see why, imagine two parties that are a large distance apart from each other, say, Alice is in Berlin and Bob in New York. While they may obtain access to shared entanglement from a third party, it is unreasonable to assume that they are able to perform global operations on the state they share. On the other hand, it is perfectly conceivable that they transmit classical messages, for example, to communicate measurement results.

The goal of this problem is to show that if Alice and Bob are in far away labs, and share a state, any measurement on Alice's part of the state can be simulated as follows: Bob performs a measurement on his side and communicates the result to Alice, who performs a local unitary transformation. This can be proven for POVMs, but for simplicity we will restrict ourselves to projective measurements.

Consider a bipartite state $|\psi\rangle_{AB}$ with Schmidt decomposition $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |a_i\rangle |b_i\rangle$ and a projective measurement $\Pi = \{\Pi_i^A\}_i$ acting on Alice's Hilbert space.

- a) Expand Π_i^A in the Schmidt basis and define a projective measurement $\Gamma = \{\Gamma_i^B\}_i$ on Bob's system such that the probability p_k^B that Bob observes result k when measuring Γ is the same as the probability p_k^A that Alice observes result k when measuring Π .

- b) Determine the post measurement states $|\phi_j^A\rangle$ after Alice measures Π and obtains result j , and $|\phi_j^B\rangle$ after Bob measures Γ and obtains result j . (both of these states are defined on the whole Hilbert space AB , the superscripts serve to identify who performed the measurement).
- c) Show that $|\phi_j^A\rangle$ and $|\phi_j^B\rangle$ are equivalent up to local unitary transformations.
- d) Describe the LOCC protocol.