Freie Universität Berlin

Tutorials on Quantum Information Theory

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Problem Sheet 9

Quantum Fourier Transform and Stabilizers

J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

1. **Quantum Fourier transform.** (9 points: 1+4+2+2) Perhaps at the heart of the majority of modern quantum algorithms lies the *phase estimation algorithm*. For this reason, it is crucial in the field of quantum computation to be familiar with phase estimation. It relies on an efficient implementation of the *quantum Fourier transform*, to which we devote this excercise.

   In classical numerics the discrete Fourier transform (DFT) is defined as the linear map $F : \mathbb{C}^N \to \mathbb{C}^N$, $x \mapsto y$ with $y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \exp\left\{\frac{2\pi \mathrm{i} jk}{N}\right\}$. The quantum Fourier transform is analogously defined as the unitary operation $\mathcal{F} : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$, $|j\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \exp\left\{\frac{2\pi \mathrm{i} jk}{2^n}\right\} |k\rangle$. (Note the identification $N = 2^n$.)

   a) Look-up the computational complexity of the fastest classical algorithm for the Fourier transform.

   The quantum Fourier transform can be implemented using the Hadamard gates H,

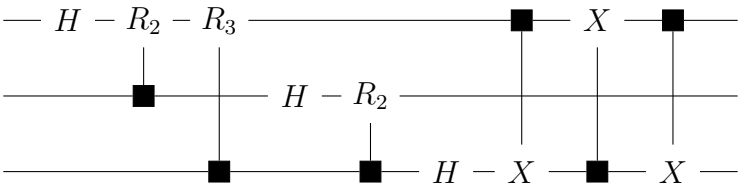
   $$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

   the controlled phase gate that applies

   $$R_k = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i}/2^k} \end{pmatrix} \tag{2}$$

   on a *target* qubit if a *control* qubit is in the state $|1\rangle$ (and the identity if the control is in $|0\rangle$) and CNOT gates. Note that in circuit diagrams controlled gates are conventionally represented by boxes on the target wires linked to dots on the control wires.

   b) Show that the following circuit implements the three qubit quantum Fourier transform

   Hint: restrict your attention to generic computational basis states as inputs.

   c) How does this generalise to the n qubit quantum Fourier transform?

   d) What is the circuit complexity of the quantum Fourier transform and how does it compare to the classical DFT algorithms?

   Note that the quantum Fourier transform can in fact be approximately implemented with only $\mathcal{O}(n \log n)$ gates [1].

[1] Cleve, Richard, and John Watrous. "Fast parallel circuits for the quantum Fourier transform." Proceedings 41st Annual Symposium on Foundations of Computer Science. IEEE, 2000.

2. **Stabilizer quantum computation.** (11 Points: 3+2+1+2+1+1+1)

One of the most celebrated results in quantum computation is a statement about the resource costs of simulating quantum computations on a classical computers. The *Gottesman-Knill theorem* states that quantum computations composed of *Clifford gates* with *stabilizer states* as inputs and a final measurement in the computational basis can be classically simulated in the sense that there exists a classical algorithm with polynomial runtime which can sample from the output distribution of such a computation. Furthermore, the so-called stabilizer formalism plays an important rôle in the development of quantum error correction.

In this problem we will trace the reasoning underlying this result. Throughout, we will let n be the number of qubits and hence $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ be the Hilbert space. Let us start with some definitions

(i) Let $G_1 = \{\pm\mathbb{1}, \pm X, \pm Y, \pm Z, \pm\mathbb{1}, \pm\mathrm{i}X, \pm\mathrm{i}Y, \pm\mathrm{i}Z\}$ be the single-qubit *Pauli group* where multiplication is the group operation.[2].

(ii) Let $G_n \coloneqq \{\bigotimes_{i=1}^n P_i, P_i \in G_1\}$ be the n-qubit Pauli group.

(iii) A *stabilizer state* is a quantum state $|\psi\rangle \in \mathcal{H}$ that is uniquely (up to a global phase) described by a set $\mathcal{S}_{|\psi\rangle} = \{S_1, \ldots, S_n\} \subset G_n$ satisfying $S_i |\psi\rangle = +1 |\psi\rangle$. We call the generalised pauli-operators S_i the stabilizers of $|\psi\rangle$.[3] We note that stabilizers are linearly independent and commutative with each others.

(iv) A Clifford operator C is a unitary on $\mathcal{H}$ which leaves G_n invariant, i.e. for all $g \in G_n$ it holds that $CgC^\dagger \in G_n$. In group theoretic slang the Clifford group $\mathcal{C} \subset U(2^n)$ is the normalizer of G_n.

Ok, now we are ready to begin.

a) Show that the set $\mathcal{S} = \{Z_1, Z_2, \ldots, Z_n\}$ uniquely stabilizes the state $|0\rangle^{\otimes n}$, where we use the notation $Z_i = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{Z}_{i\text{-th qubit}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ for the operator acting as Z on the i-th qubit and as the identity on all other qubits.

b) Show that n stabilizers suffice to uniquely characterize an arbitrary state in the *Clifford orbit* of $|0\rangle^{\otimes n}$, that is the states $|\psi\rangle$ for which there exists a (unique) Clifford operator C such that $|\psi\rangle = C |0\rangle^{\otimes n}$.

c) Give a stabilizer representation of $|+\rangle \otimes |0\rangle \otimes |-\rangle$.

Any Clifford operator can be expressed as a product of single- and two-qubit Clifford operators, and indeed as a product from the generating set $\{CNOT, H, S\}$, where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

d) Show that this gate set is sufficient to generate all Pauli matrices starting from any single-qubit Pauli matrix.

e) Argue that one can efficiently (in the number of qubits and gates) determine the stabilizer set of a state generated by a (known) Clifford circuit (comprising $CNOT, H, S$ gates) applied to a stabilizer state.

From the above reasoning, we conclude that we can efficiently simulate the effect of a Clifford circuit applied to a stabilizer state by keeping track of the stabilizers.

Now, let us assume that we measure the first qubit in the Z basis.

[2] Convince yourself that G_1 is closed under multiplication and the unsigned Pauli matrices are not.

[3] More generally, we can talk about subspaces stabilized by a set $\mathcal{S} \subset G_n$. This is a key insight in the theory of error correction codes.

f) Assume Z_1 commutes with all stabilizers. What is the probability of obtaining outcome $+1$?

One can show that in case Z_1 does not commute with all stabilizers, one can find an alternative set of stabilizers such that it anti-commutes with one of them but commutes with all remaining ones.

g) Use the existence of such a stabilizer to show that the measurement outcome is uniformly random. What is the post-measurement state?

In fact, this generalizes to the measurement of an arbitrary Pauli operator $g \in G_n$. Therefore, we see that checking commutation with the stabilizers gives us a recipe for efficiently simulating samples resulting from computational basis measurements.