Exercise Sheet 0: Elements of Linear Algebra

This exercise sheet serves as a warm-up for the course. We want to familiarize ourselves with the most important tool we need in quantum information: Linear Algebra in bra-ket notation.

Bra-ket notation

Introduction. Bra-ket notation is a different way of writing up vectors that physicists usually find more appealing. There is not much magic going on. Normally, vectors in $\mathbb{C}^d$ are written by using bold symbols $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^d$ and we can denote inner products (scalar products) as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{d} u_i^* v_i = \boldsymbol{u}^\dagger \boldsymbol{v}. \tag{1}$$

Note here that we need to do a complex conjugation of the left vector because we are dealing with complex numbers. In bra-ket notation, we just replace the symbol of a (column) vector with a *ket* $\boldsymbol{v} \to |v\rangle$ and of a conjugated (row) vector with a *bra* $\boldsymbol{u}^\dagger = \langle u|$. Putting both together we close the *bracket* to obtain the inner product again

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle u|v\rangle = \sum_{i=1}^{d} u_i^* v_i. \tag{2}$$

We will also use the conjugate transpose to switch between bra and ket:

$$|v\rangle^\dagger = \langle v| \tag{3}$$

$$\langle v|^\dagger = |v\rangle\,. \tag{4}$$

In the following, we will also make use of *outer* products that describe maps, i.e. matrices. Consider for example the outer products

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

The outer product of two vectors is normally written as $\boldsymbol{u}\boldsymbol{v}^\dagger$ and in bra-ket notation becomes $\boldsymbol{u}\boldsymbol{v}^\dagger \to |u\rangle\langle v|$.

Last but not least we will need to multiply vectors with matrices, for a given matrix A usually written as $A\boldsymbol{u}$. This becomes $A\boldsymbol{u} \to A|u\rangle$. If we take the inner product of a vector $\boldsymbol{u}$ with $A\boldsymbol{v}$ we obtain

$$\langle \boldsymbol{u}, A\boldsymbol{v} \rangle = \langle u|A|v\rangle. \tag{7}$$

Note that if we take the conjugate (bra) of a matrix-vector product we have to use the conjugate transpose

$$(A\,|v\rangle)^\dagger = \langle v|\,A^\dagger. \tag{8}$$

9 P. **Exercise 1** (Warmup)**.** Let us denote with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

the two basis vectors of $\mathbb{C}^2$.

2 P. (a) Write the following vectors in bra-ket notation:

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} 3-i \\ 1+i \end{pmatrix} \qquad \begin{pmatrix} i & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 \end{pmatrix} \tag{10}$$

Solution

$$\begin{aligned} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} &= 1/\sqrt{2}(|0\rangle + |1\rangle) \\ \begin{pmatrix} 3-i \\ 1+i \end{pmatrix} &= (3-i)\,|0\rangle + (1+i)\,|1\rangle \\ \begin{pmatrix} i & 3 \end{pmatrix} &= i\,\langle 0| + 3\,\langle 1| \\ \begin{pmatrix} 0 & 2 \end{pmatrix} &= 2\,\langle 1| \end{aligned}$$

3 P. (b) Write the following matrices in bra-ket notation:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

Solution

$$\begin{aligned} X &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ Y &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned}$$

2 P. (c) Compute $\langle 0|X|0\rangle$ by explicitly writing down the vectors.

Solution

$$\begin{aligned} \langle 0|X|0\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 0. \end{aligned}$$

2 P. (d) Now compute (or argue) the values of $\langle 0|0\rangle$, $\langle 1|1\rangle$ and $\langle 0|1\rangle$. Then calculate $\langle 0|X|0\rangle$ in bra-ket notation.

Solution

$$\begin{aligned} \langle 0|X|0\rangle &= \langle 0|\big(0\rangle\langle 1| + |1\rangle\langle 0|\big)|0\rangle \\ &= \underbrace{\langle 0|0\rangle}_{=1}\underbrace{\langle 1|0\rangle}_{=0} + \underbrace{\langle 0|1\rangle}_{=0}\underbrace{\langle 0|0\rangle}_{=1} \\ &= 0 \end{aligned}$$

9 P. Exercise 2 (Abstract calculations)**.** We now look at the abstract vector space $\mathbb{C}^d$.

1 P. (a) For a ket vector in $\mathbb{C}^d$ given in the computational basis as

$$|v\rangle = \sum_{i=1}^{d} v_i \, |i\rangle \, , \tag{12}$$

write down its associated bra $\langle v|$.

Solution

$$\langle v| = \sum_{i=1}^{d} v_i^* \, \langle i| \, .$$

2 P. (b) Write a matrix $A \in \mathbb{C}^{d\times d}$ with entries A_{ij} in the computational basis in bra-ket notation.

Solution

$$A = \sum_{i=1}^{d} \sum_{j=1}^{d} A_{ij} \, |i\rangle\langle j| .$$

2 P. (c) A matrix A has eigenvalues $\{a_i\}_{i=1}^d$ with associated eigenvectors $\{|a_i\rangle\}_{i=1}^d$. Give the expansion of A in terms of its eigenvectors in bra-ket notation.

Solution

$$A = \sum_{i=1}^{d} a_i \, |a_i\rangle\langle a_i| .$$

4 P. (d) Let the matrix A have a singular value decomposition $A = UDV^\dagger$ with unitary matrices U and V and a diagonal matrix D. Express this decomposition of A in bra-ket notation.

Solution

The matrix D is diagonal in the computational basis, hence we can write $D = \sum_{i=1}^d d_i |i\rangle\langle i|$ where $\{d_i\}_{i=1}^d$ are the singular values of A. We expand

$$\begin{aligned} A &= UDV^\dagger \\ &= U \left(\sum_{i=1}^{d} d_i |i\rangle\langle i| \right) V^\dagger \\ &= \sum_{i=1}^{d} d_i \, U|i\rangle\langle i|V^\dagger . \end{aligned}$$

Now let us define the vectors $|u_i\rangle = U\,|i\rangle$ and $|v_i\rangle = V\,|i\rangle$. Then we have the form

$$A = \sum_{i=1}^{d} d_i \, |u_i\rangle\langle v_i| .$$

3 P. **Bonus Exercise 1.** A ket $|v\rangle$ represents an element of a Hilbert space (a vector space with scalar product). Find out and describe what a bra $\langle v|$ represents in abstract terms.

Solution

As a ket represents a vector in $\mathbb{C}^d$, a bra represents a vector in the *dual space* of $\mathbb{C}^d$, which is the space of *linear maps* from $\mathbb{C}^d$ to $\mathbb{C}$. Any such map can be constructed as a linear combination of bra vectors. To learn more about this, google a bit.

Resolution of the identity. Another simple fact of linear algebra that can be quite useful is the *resolution of the identity*, i.e. the fact that we can write the identity matrix $\mathbb{I} \in \mathbb{C}^{d\times d}$ as

$$\mathbb{I} = \sum_{i=1}^{d} |a_i\rangle\langle a_i| \tag{13}$$

for any orthonormal basis $\{|a_i\rangle\}_{i=1}^{d}$.

4 P. **Exercise 3** (Resolution of the identity).

3 P. (a) Use the resolution of the identity to determine the bra-ket form of an arbitrary matrix A. (Compare to Exercise 2 (b))

Solution

$$\begin{aligned} A &= \mathbb{I}A\mathbb{I} \\ &= \left(\sum_{i=1}^{d} |i\rangle\langle i|\right) A \left(\sum_{j=1}^{d} |j\rangle\langle j|\right) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} \langle i|A|j\rangle\, |i\rangle\langle j|. \end{aligned}$$

1 P. (b) Compare your result to Exercise 2 (b) and give the formula for the entries of the matrix A_{ij} relative to the computational basis.

Solution

$$A_{ij} = \langle i|A|j\rangle.$$

Tensor products

Introduction. When multiple quantum systems are combined, their mathematical description needs the so-called *tensor product*. We will study this from the perspective of linear algebra. Let us consider two Hilbert spaces $\mathbb{C}^d$ and $\mathbb{C}^{d'}$ with orthonormal basis systems $\{|i\rangle\}_{i=1}^{d}$ and $\{|j\rangle\}_{j=1}^{d'}$ respectively. On an intuitive level, the tensor product $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ of the two spaces aims to describe *all possible combinations* of vectors in $\mathbb{C}^d$ and $\mathbb{C}^{d'}$. We can construct an orthonormal basis for the new space $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ by combining the basis elements of both spaces into a new basis $\{|i\rangle \otimes |j\rangle\}_{i=1}^{d}{}_{j=1}^{d'}$. This means we can expand any vector in the tensor product space as

$$|v\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} \Rightarrow |v\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d'} v_{ij}(|i\rangle \otimes |j\rangle). \tag{14}$$

Note that so far we just made up a new vector space with a new basis that has two indices instead of one. One can reduce this to one index by making up a new basis $|k\rangle = |i\rangle \otimes |j\rangle$ where $k = i + (j-1)\cdot d$, which is something people often do when doing numerical implementations.

To complete the definition of the tensor product space, we also need a way to take vectors $|u\rangle \in \mathbb{C}^d$ and $|v\rangle \in \mathbb{C}^{d'}$ and combine them to a vector in the product space. Let $|u\rangle = \sum_{i=1}^{d} u_i |i\rangle$ and $|v\rangle = \sum_{j=1}^{d'} v_j |j\rangle$. Then, the tensor product operation is naturally obtained by treating the operation $\otimes$ like a product:

$$
\begin{aligned}
|u\rangle \otimes |v\rangle &= \left(\sum_{i=1}^{d} u_i |i\rangle\right) \otimes \left(\sum_{j=1}^{d'} v_j |j\rangle\right) && (15)\\
&= \sum_{i=1}^{d}\sum_{j=1}^{d'} (u_i |i\rangle \otimes v_j |j\rangle) && (16)\\
&= \sum_{i=1}^{d}\sum_{j=1}^{d'} u_i v_j (|i\rangle \otimes |j\rangle). && (17)
\end{aligned}
$$

All definitions above extend in a similar way to tensor products of bras, i.e. expressions like $\langle u|\otimes\langle v| = (|u\rangle\otimes|v\rangle)^\dagger$. In quantum information science, people are usually quite lazy. Therefore, you will often see the abbreviation $|i\rangle \otimes |j\rangle = |i,j\rangle = |ij\rangle$.

4 P. **Exercise 4** (Tensor products of vectors)**.**

1 P. (a) What is the dimension of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$?

Solution

It is $d \times d'$.

3 P. (b) For a given vector $|v\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$, give a formula for the coefficients v_{ij} of Eq. (14). (Hint: what is $(\langle i| \otimes \langle j|)(|i'\rangle \otimes |j'\rangle) = \langle i,j|i',j'\rangle$?).

Solution

We have

$$v_{ij} = (\langle i| \otimes \langle j|)\,|v\rangle = \langle i,j|v\rangle$$

because

$$
\begin{aligned}
(\langle i| \otimes \langle j|)\,|v\rangle &= (\langle i| \otimes \langle j|) \sum_{i'=1}^{d}\sum_{j'=1}^{d'} v_{i'j'}(|i'\rangle \otimes |j'\rangle)\\
&= \sum_{i'=1}^{d}\sum_{j'=1}^{d'} v_{i'j'}(\langle i| \otimes \langle j|)(|i'\rangle \otimes |j'\rangle)\\
&= \sum_{i'=1}^{d}\sum_{j'=1}^{d'} v_{i'j'}(\langle i|i'\rangle \otimes \langle j|j'\rangle)\\
&= \sum_{i'=1}^{d}\sum_{j'=1}^{d'} v_{i'j'}\delta_{ii'}\delta_{jj'}\\
&= v_{ij}.
\end{aligned}
$$

Kronecker product. Next, we will consider a particular way to find explicit vectors representing the basis of the tensor product space. Let us consider again $\mathbb{C}^2$ with the basis vectors

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \qquad (18)$$

We can give new basis vectors for $\mathbb{C}^2 \otimes \mathbb{C}^2$ by using the *Kronecker product*, which can be understood as follows:

$$|u\rangle \otimes |v\rangle \to \begin{pmatrix} u_1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ u_2 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \end{pmatrix}. \tag{19}$$

We did not write equals signs because we remove the parentheses in the middle step. Using the above product, we obtain an explicit basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$\begin{aligned} |0\rangle \otimes |0\rangle = |0,0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |0\rangle \otimes |1\rangle = |0,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ |1\rangle \otimes |0\rangle = |1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |1\rangle \otimes |1\rangle = |1,1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{20}$$

The construction extends to products of larger spaces in the natural way.

2 P. **Exercise 5** (Kronecker product of vectors)**.** We use the setting of $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the Kronecker product described above. Compute the following tensor products in the Kronecker product basis:

$$(1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) \otimes (1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) \tag{21}$$
$$(1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) \otimes (1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle) \tag{22}$$
$$(1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle) \otimes (1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) \tag{23}$$
$$(4i\,|0\rangle - 3\,|1\rangle) \otimes (|0\rangle - i\,|1\rangle). \tag{24}$$

Solution

$$(1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) \otimes (1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$(1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) \otimes (1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle) = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$(1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle) \otimes (1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle) = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$$

$$(4i\,|0\rangle - 3\,|1\rangle) \otimes (|0\rangle - i\,|1\rangle) = \begin{pmatrix} 4i \\ 4 \\ -3 \\ 3i \end{pmatrix}.$$

Tensor products of matrices. We have discussed how we can construct tensor products of vectors. Next we care about tensor products of matrices acting on these vectors. We can define the tensor product of two operations through the rather sensible equation

$$(A\,|u\rangle) \otimes (B\,|v\rangle) = (A \otimes B)(|u\rangle \otimes |v\rangle). \tag{25}$$

11 P. **Exercise 6** (Tensor products of matrices)**.**

2 P. (a) Use the definition of Eq. (25) to show that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Solution

By definition, we have that

$$\begin{aligned}(A \otimes B)(C \otimes D)(|u\rangle \otimes |v\rangle) &= (A \otimes B)(C\,|u\rangle \otimes D\,|v\rangle)\\ &= (AC\,|u\rangle \otimes BD\,|v\rangle)\\ &= (AC \otimes BD)(|u\rangle \otimes |v\rangle).\end{aligned}$$

4 P. (b) For two matrices $A = \sum_{i=1}^{d}\sum_{i'=1}^{d} A_{ii'}|i\rangle\langle i'|$ and $B = \sum_{j=1}^{d'}\sum_{j'=1}^{d'} B_{jj'}|j\rangle\langle j'|$, determine the bra-ket expansion of $A \otimes B$.

Solution

We know that any matrix C can be expressed in an orthonormal basis $\{|v_i\rangle\}_{i=1}^{d}$ as

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\langle i|C|j\rangle\,|i\rangle\langle j|.$$

Therefore, the action of C is completely determined by knowing all the terms $\langle i|C|j\rangle$ in some basis. In the case of $A \otimes B$, this means it suffices to calculate

$$\begin{aligned}&(\langle i| \otimes \langle j|)(A \otimes B)(|i'\rangle \otimes |j'\rangle)\\ &= (\langle i| \otimes \langle j|)(A\,|i'\rangle) \otimes (B\,|j'\rangle)\\ &= (\langle i| \otimes \langle j|)\left(\sum_{i''=1}^{d}\sum_{i'''=1}^{d} A_{i''i'''}|i''\rangle\langle i'''|i'\rangle\right) \otimes \left(\sum_{j''=1}^{d'}\sum_{j'''=1}^{d'} B_{j''j'''}|j''\rangle\langle j'''|j'\rangle\right)\\ &= (\langle i| \otimes \langle j|)\left(\sum_{i''=1}^{d} A_{i''i'}|i''\rangle\right) \otimes \left(\sum_{j''=1}^{d'} B_{j''j'}|j''\rangle\right)\\ &= A_{ii'}B_{jj'}.\end{aligned}$$

Therefore, $A \otimes B$ can be written as

$$\begin{aligned}A \otimes B &= \sum_{i,i'=1}^{d}\sum_{j,j'=1}^{d'} A_{ii'}B_{jj'}(|i\rangle \otimes |j\rangle)(\langle i'| \otimes \langle j'|)\\ &= \sum_{i,i'=1}^{d}\sum_{j,j'=1}^{d'} A_{ii'}B_{jj'}(|i\rangle\langle i'| \otimes |j\rangle\langle j'|)\\ &= \sum_{i,i'=1}^{d}\sum_{j,j'=1}^{d'} A_{ii'}B_{jj'}|i,j\rangle\langle i',j'|.\end{aligned}$$

2 P. (c) Compute the matrix form of the operator $X \otimes X$ in the Kronecker product basis. (Hint: Use the fact that $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$.

Solution

Because the basis vectors of the Kronecker product basis only map to each other, the matrix representation of $X \otimes X$ consists of column vectors with only one 1 and three zeros. We are left to track down exactly where they are which can easily seen by taking the action of $X \otimes X$ on the computational basis:

$$(X \otimes X)(|0\rangle \otimes |0\rangle) = |1\rangle \otimes |1\rangle$$
$$(X \otimes X)(|0\rangle \otimes |1\rangle) = |1\rangle \otimes |0\rangle$$
$$(X \otimes X)(|1\rangle \otimes |0\rangle) = |0\rangle \otimes |1\rangle$$
$$(X \otimes X)(|1\rangle \otimes |1\rangle) = |0\rangle \otimes |0\rangle .$$

Therefore, the matrix representation of $X \otimes X$ is

$$X \otimes X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

3 P. (d) Look up the multiplication rule for the Kronecker product $A \otimes B$ of two matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{26}$$

and write it down. Then, compute the following matrix tensor products in the Kronecker product basis:

$$X \otimes Y, Z \otimes X. \tag{27}$$

Here, X, Y, Z are as defined in Exercise 1.

Solution

The multiplication rule is given by

$$A \otimes B \to \begin{pmatrix} A_{11}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{12}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ A_{21}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{22}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{pmatrix}$$

$$\to \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}.$$

We recall from Exercise 1 the definitions

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With these, we can compute:

$$X \otimes Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$Z \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

3 P. **Bonus Exercise 2.** Are there vectors $|v\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ that you *cannot* write as $|v\rangle = |u\rangle \otimes |v\rangle$ for $|u\rangle, |v\rangle \in \mathbb{C}^2$?

Solution

Yes, there are such vectors! You will revisit this topic soon under the name of *quantum entanglement*. An example is

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle).$$

Total Points: 39 (+6)