Exercise Sheet 1: Basics of Quantum Information Theory

This exercise sheet tries to teach you some of the basics of quantum information theory.

Density Matrix Formalism

Introduction. There are multiple ways to approach a description of quantum mechanics from the analytical point of view. Quantum mechanics courses usually rely mostly on the formalism of quantum states, normally expressed as $|\psi\rangle$, and the Schrödinger equation. For our purposes, we will, however, mostly use the *density matrix formulation* of quantum mechanics, which allows for a simpler treatment of *probabilistic mixtures* of quantum states. These arise for example when a quantum system undergoes unwanted random interactions with an environment, introducing "noise" to a quantum state.

The density matrix formulation starts from the following (incomplete) set of postulates:

(I) Each physical system is associated with a Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$. The **(mixed) state** of a quantum system is described by a non-negative (also called positive semi-definite, this means that all eigenvalues of the matrix are non-negative), self-adjoint linear operator with unit trace, i.e. an element of[1]

$$\mathcal{D} := \{\rho \in L(\mathcal{H}) \mid \rho = \rho^\dagger,\ \rho \geq 0,\ \mathrm{Tr}[\rho] = 1\}. \tag{1}$$

Here, $\rho \geq 0$ is the notation we use to say that ρ is positive semi-definite.

(II) **Observables** are represented by Hermitian operators on $\mathcal{H}$. The expectation value of an observable A in the state ρ is given by $\langle A\rangle_\rho = \mathrm{Tr}[A\rho]$.

(III) The **time-evolution** of the state of a quantum system satisfies

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H, \rho],$$

where H is the *Hamiltonian*, the observable associated to the total energy of the system.

2 P. **Exercise 1** (Rank one projectors)**.** Show that the set

$$\mathcal{P} = \{\pi \in L(\mathcal{H}) \mid \pi = \pi^\dagger,\ \pi^2 = \pi,\ \mathrm{rank}\,\pi = 1\} \tag{2}$$

of orthogonal projectors onto one-dimensional subspaces of $\mathcal{H}$ is a subset of $\mathcal{D}$.

Most probably, you have originally learned another definition for quantum states in your first quantum mechanics course. Namely, **pure quantum states** are rays of the Hilbert space $\mathcal{H}$. The rays of a Hilbert space are the equivalence classes of unit vector that only differ by a phase factor. In symbols, we have $\mathrm{rays}(\mathcal{H}) = \{|\psi\rangle \in \mathcal{H} \mid \||\psi\rangle\|_2^2 = 1\}/\sim$. The equivalence relation $|\psi\rangle \sim |\phi\rangle$ captures the fact that if there exist $\alpha \in \mathbb{R}$ such that $|\psi\rangle = \mathrm{e}^{i\alpha}|\phi\rangle$, then $|\psi\rangle$ and $|\phi\rangle$ represent the same state. Often physicists tend to drop the equivalence relation and talk about unit vectors as quantum states instead of rays.

5 P. **Exercise 2** (Pure states)**.** In this exercise, we will show that the set $\mathcal{P}$ of Eq. (2) is equivalent to the set of pure quantum states and use this to derive the time-evolution in the density matrix formalism from the pure state Schrödinger equation.

[1] In quantum information theory, it will be sufficient to consider finite-dimensional Hilbert spaces most of the time. A finite-dimensional Hilbert space is simply a vector space. In infinite dimension there are more subtleties, but these do not concern us.

2 P. (a) Show that the mapping

$$[|\psi\rangle] \mapsto |\psi\rangle\langle\psi| \tag{3}$$

is a bijection between the set $\mathcal{P}$ defined in Eq. (2) and rays($\mathcal{H}$) irrespective of which representative of $[|\psi\rangle]$ is chosen.

1 P. (b) Show that the mapping of Eq. (3) is the correct one as it reproduces the same expectation values for any observable A.

2 P. (c) Starting from the Schrödinger equation for pure states, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = -iH|\psi\rangle \tag{4}$$

derive the corresponding evolution equation for density matrices

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H, \rho]. \tag{5}$$

Here $[A, B] = AB - BA$ is the commutator of two matrices. (Hint: start by proving this for $\rho = \pi$ a pure state, then use linearity.)

We have seen that density matrices describe both mixed and pure quantum states. Let us define the following function of the state

$$f : \mathcal{D} \to \mathbb{R}, \rho \mapsto \mathrm{Tr}[\rho^2]. \tag{6}$$

Before we come to the next exercise, we will also introduce the so-called *Hilbert-Schmidt inner product*, which endows the space of matrices with an inner product. It is defined as

$$\langle A, B\rangle := \mathrm{Tr}[A^\dagger B]. \tag{7}$$

As any proper inner product, it also obeys a Cauchy-Schwarz inequality:

$$\mathrm{Tr}[A^\dagger B]^2 \leq \mathrm{Tr}[A^\dagger A]\,\mathrm{Tr}[B^\dagger B]. \tag{8}$$

9 P. Exercise 3 (Purity)**.**

3 P. (a) Show that $\frac{1}{d} \leq f(\rho) \leq 1$, where d is the dimension of the Hilbert space $\mathcal{H}$. (Hint: Use the Cauchy-Schwarz inequality)

3 P. (b) Show that $f(\rho) = 1$ if and only if ρ is a pure state.

3 P. (c) What state attains the lower bound $f(\rho) = \frac{1}{d}$? Argue that $f(\rho)$ can be seen as a measure of "purity" of the state ρ.

5 P. Exercise 4 (Decompositions of mixed states)**.**

2 P. (a) Show that every mixed state of a finite-dimensional quantum system can be written as a convex decomposition of pure states.

2 P. (b) Consider the following two (macroscopically different) preparation schemes of a large number of polarised photons:

Preparation A. For each photon we toss a fair coin. Depending on whether we get head or tail, we prepare the photon to have either vertical or horizontal *linear* polarisation.

Preparation B. For each photon we toss a fair coin. Depending on whether we get head or tail, we prepare the photon to have either left-handed or right-handed *circular* polarisation.

Note: You can simply think of the polarization of the light as a binary variable and of the polarization axis as a local basis. That is, the vertical and horizontal linear polarizations may be identified with the $|0\rangle$ and $|1\rangle$ eigenstates of the Z operator. Likewise you may interpret the left- and right handed circular polarizations as the $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ eigenstates of the X operator.

Write down the density matrices $\rho_A^{(m)}$ and $\rho_B^{(m)}$ describing the mixed quantum states obtained after m rounds of Preparations A and B, respectively.

1 P. (c) Use the result of (b) to argue that, having only access to the photons produced by the preparation procedures in (b), we cannot distinguish whether Preparation A or Preparation B was used.

4 P. Bonus Exercise 1. Argue that if it were possible to distinguish Preparation A from Preparation B (from the previous exercise) by measuring the photons, then this could be used to communicate a bit of information without actually sending any (classical or quantum) information carrier. Which fundamental physical principle would this violate? (Hint: What is the reduced state of the maximally entangled state $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$?)

Composite Quantum Systems

Next, we will see that the generalization to density matrices is a necessary one if we want to study subsystems. Consider a bipartite system AB with Hilbert space $\mathcal{H} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and an observable that is only supported on the subsystem A as $O_A \otimes \mathbb{I}_B$. We will see that the restriction to a subsystem is described by the *partial trace*: For a a linear operator (matrix) $M\colon \mathcal{H} \to \mathcal{H}$ on the composite system AB, the partial trace with respect to the system B is defined as

$$\mathrm{Tr}_B[M] = \sum_{j=1}^{d_B} (\mathbb{I}_A \otimes \langle j|_B) M (\mathbb{I}_A \otimes |j\rangle_B), \tag{9}$$

where $\{|j\rangle_B\}$ is an arbitrary orthonormal basis for $\mathbb{C}^{d_B}$ (as with the trace, this definition is independent of the particular choice of the basis). In quantum information theory, we usually say "we trace out the system B".

10 P. Exercise 5.

2 P. (a) As a technical prerequisite, prove that a self-adjoint operator is positive semi-definite, i.e. has only non-negative eigenvalues, if and only if $\langle v|M|v\rangle \geq 0$ for all $|v\rangle$.

3 P. (b) Show that the partial trace of a state with respect to the system B (density operator) is a valid state on the subsystem A.

2 P. (c) Prove that for any state ρ_{AB} we have

$$\mathrm{Tr}[\rho_{AB}(O_A \otimes \mathbb{I}_B)] = \mathrm{Tr}[\mathrm{Tr}_B[\rho_{AB}] O_A]. \tag{10}$$

for all observables O_A. That is, the partial trace is the *reduced state* on the subsystem A.

3 P. (d) Reduced states of pure states are not necessarily pure. Let $d_A = d_B = d$. Show that there is no pure state $|\psi_A\rangle\langle\psi_A|$ acting on A that satisfies

$$\mathrm{Tr}[\rho_{AB}(O_A \otimes \mathbb{I}_B)] = \mathrm{Tr}[|\psi_A\rangle\langle\psi_A| O_A] \tag{11}$$

for $\rho_{AB} = |\Omega_{AB}\rangle\langle\Omega_{AB}|$ and all observables O_A. Here,

$$|\Omega\rangle := d^{-\frac{1}{2}} \sum_{j=1}^{d} |j, j\rangle \tag{12}$$

is the *maximally entangled state*.

Total Points: 30 (+4)