Exercise Sheet 3: Teleportation and Co.

Teleportation

In the lecture, you have seen the state teleportation protocol that allowed Alice to send an arbitrary quantum state to Bob, requiring that Alice and Bob share a maximally entangled state and Alice sends Bob two classical bits.

1 P. **Exercise 1.** Make a schematic drawing of the quantum state teleportation protocol.

We wish to use this sheet to construct some further variants of the teleportation protocol. For this, we first have to improve our understanding of the Bell states.

2 P. **Exercise 2.** Recall the Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{1}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{2}$$

Show that you can interconvert the Bell states through local operations only, i.e. through operations that only act on one of the two systems at a time.

Let us now come to our first variation, where we restrict the classes of states that Alice wants to send.

4 P. **Exercise 3.** We now look at a restricted scenario and assume that both Alice and Bob know that the state that Alice wants to transmit is on the equator of the Bloch sphere, i.e. one of the states $|\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\alpha}|1\rangle)$ for $0 \leq \alpha \leq 2\pi$. Moreover, Alice knows the value α, i.e. she prepares the state herself. In this case, we can actually reduce the amount of classical communication between Alice and Bob.

2 P. (a) Let us define $|\alpha^\perp\rangle = |\alpha + \pi\rangle$. Show that $|\alpha^\perp\rangle$ is orthogonal to $|\alpha\rangle$ and express the computational basis states $|0\rangle$ and $|1\rangle$ in terms of $|\alpha\rangle$ and $|\alpha^\perp\rangle$.

2 P. (b) Alice and Bob now devise the following protocol: As in the regular teleportation protocol, they share a maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice measures her half of this previously shared state in the basis $\{|\alpha\rangle, |\alpha^\perp\rangle\}$. Show that communicating the outcome of this measurement (one bit) is sufficient for Bob to faithfully obtain the state $|\alpha\rangle$.

1 P. **Exercise 4.** There is also another way Alice and Bob can reduce the amount of classical communication required in the original state teleportation protocol to a single bit when they accept that the protocol does not always succeed. How can they do it while still knowing if the protocol succeeded?

1 P. **Exercise 5.** Use the state teleportation protocol to construct a new protocol that allows the following. At the beginning of the experiment, Bob holds a state $|\psi\rangle$ and at the end holds $U|\psi\rangle$ where U was applied by Alice. The protocol should require two maximally entangled states and four bits of classical communication.

3 P. **Bonus Exercise 1.** Show that in the protocol of the previous exercise, if both Alice and Bob know that the operation U is one of $R_Z(\alpha) = e^{-i\alpha Z/2}$ (a Pauli-Z rotation) for $0 \leq \alpha \leq 2\pi$, then only three classical bits are required. (*Hint:* Establish the fact that $[Z, R_Z(\alpha)] = 0$ for all $0 \leq \alpha \leq 2\pi$. Use this to reason about the Z corrections involved in the previous exercise)

Let us now come to the last variant of teleportation we want to consider.

3 P. **Exercise 6.** Let us now consider the following situation: Both Alice and Bob each share a maximally entangled state with a third party, Charlie, meaning the system state is $|\Phi^+\rangle_{AC_1}|\Phi^+\rangle_{BC_2}$.

Then, Charlie takes his two qubits and measures them in the Bell basis (compare Exercise 2). Show that it is sufficient for Charlie to communicate two bits so that at the end Alice and Bob share the state $|\Phi^+\rangle_{AB}$ (*Hint:* Use the result of Exercise 2). What can this protocol be used for?

Schatten norms

The next part of the sheet is dedicated to Schatten p-norms, a very important mathematical concept that we use a lot in this course. One way to define the Schatten p-norm with $p \in [1, \infty)$ for a matrix $A \in \mathbb{C}^{n\times n}$ is

$$\|A\|_p := \left(\mathrm{Tr}\left[|A|^p\right]\right)^{\frac{1}{p}}, \tag{3}$$

where $|A| := \sqrt{A^\dagger A}$ is the matrix absolute value. Note that here, $A^\dagger A \geq 0$ by construction, which means that the unique positive square root exists and $\sqrt{A^\dagger A}$ is well defined and positive semi-definite. Furthermore, the case $p = \infty$ is defined as the limit

$$\|A\|_\infty := \lim_{p\to\infty} \|A\|_p. \tag{4}$$

For the next exercise, we recall the definition of the p-norm for vectors $\boldsymbol{v} = (v_1, v_2, \ldots, v_d)$

$$\|\boldsymbol{v}\|_p = \left(\sum_{i=1}^{d} |v_i|^p\right)^{\frac{1}{p}}. \tag{5}$$

3 P. **Exercise 7.** In this exercise, we will relate the Schatten p norm of a matrix to the vector p norm of its vector of singular values. To do so, we first need some prerequisites.

1 P. (a) Let $B = UDU^\dagger$ a diagonalizable matrix, and let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex-valued function with series expansion $f(x) = \sum_{k=0}^{\infty} f_k x^k$, with $f_k \in \mathbb{C}$ for each $k \in \mathbb{N}$. Then, we can define an analogous function f acting on matrices as

$$f(X) = \sum_{k=0}^{\infty} f_k X^k, \tag{6}$$

defined as an analogous series with the same coefficients. Show that

$$f(B) = f(UDU^\dagger) = Uf(D)U^\dagger, \tag{7}$$

i.e. applying a function to a diagonalizable matrix is like applying the function to the eigenvalues of said matrix.

2 P. (b) Let A be a Hermitian matrix and let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be the vector of its singular values. Show that

$$\|A\|_p = \|\boldsymbol{\lambda}\|_p \tag{8}$$

for all p. (*Hint*: you can apply the result of the previous exercise to $f(X) = \sqrt{X}$ and $f(X) = X^p$.)

2 P. **Exercise 8.** Show that for a given vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_d)$

$$\|\boldsymbol{v}\|_\infty := \lim_{p\to\infty} \|\boldsymbol{v}\|_p = \max_{1\leq i\leq d} |v_i|.$$

What can we conclude from this about $\|A\|_\infty$?

Next, we wish to establish some properties of the Schatten p norms.

6 P. **Exercise 9.**

1 P. (a) Show that for all unitaries U and V,

$$\|UAV^\dagger\|_p = \|A\|_p. \tag{9}$$

1 P. (b) Show that the Schatten p norms tensorize, i.e. that

$$\|A \otimes B\|_p = \|A\|_p \|B\|_p. \tag{10}$$

2 P. (c) Prove the following variational formulation for the Schatten p-norm

$$\|A\|_p = \sup_{\|X\|_q = 1} \mathrm{Tr}[AX] \tag{11}$$

for $1/p + 1/q = 1$. You can make use of the variational formulation of the vector p norm

$$\|\boldsymbol{v}\|_p = \sup_{\|\boldsymbol{z}\|_q = 1} \langle \boldsymbol{v}, \boldsymbol{z} \rangle. \tag{12}$$

2 P. (d) Show that the Schatten norms are ordered, i.e. if $p \leq q$ then

$$\|A\|_p \geq \|A\|_q. \tag{13}$$

Last but not least, we want to observe two important applications for quantum information.

2 P. Exercise 10.

1 P. (a) Show that $\|\rho\|_1 = 1$ for any quantum state ρ.

1 P. (b) Show that $\|\rho\|_2^2$ is the purity of the state.

There are some rather important inequalities that we will need later in the course which we establish as a bonus exercise. The next exercise gives a matrix generalization of Hölder's inequality for vectors which posits

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle \leq \langle |\boldsymbol{v}|, |\boldsymbol{w}| \rangle \leq \|\boldsymbol{v}\|_p \|\boldsymbol{w}\|_q \tag{14}$$

for $1/p + 1/q = 1$ and $|\boldsymbol{v}|$ being the vector $\boldsymbol{v}$ where all entries are replaced by their absolute values.

2 P. Bonus Exercise 2. Hölder's inequality for Schatten norms takes the form

$$\mathrm{Tr}[A^\dagger B] \leq \|A^\dagger B\|_1 \leq \|A\|_p \|B\|_q \tag{15}$$

for any $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$. Prove this inequality. If we use the notation $\boldsymbol{\sigma}^\downarrow(A)$ to denote the vector of singular values of A in descending order, then you can use the following inequality:

$$\sum_{i=1}^d \sigma_i^\downarrow(AB) \leq \sum_{i=1}^d \sigma_i^\downarrow(A)\sigma_i^\downarrow(B). \tag{16}$$

2 P. Bonus Exercise 3. A different class of matrix norms is given by the *induced norms* which are defined via the vector p norms. We define

$$\|A\|_{p\to q} := \sup_{\|\boldsymbol{v}\|_p = 1} \|A\boldsymbol{v}\|_q. \tag{17}$$

These norms are useful, because they allow us to establish inequalities of the form

$$\|A\boldsymbol{v}\|_q \leq \|A\|_{p\to q} \|\boldsymbol{v}\|_p. \tag{18}$$

Show that

$$\|A\|_{2\to 2} = \|A\|_\infty. \tag{19}$$

2 P. **Bonus Exercise 4.** Another important property of norms is submultiplicativity. First, establish that

$$\|AB\|_p \leq \|A\|_\infty \|B\|_p \tag{20}$$

and then conclude that

$$\|AB\|_p \leq \|A\|_p \|B\|_p. \tag{21}$$

You can make use of the min-max-principle for singular values which posits that

$$\sigma_i^\uparrow(A) = \min_{S \subseteq \mathbb{C}^d, \dim(S)=i} \max_{\boldsymbol{v} \in S, \|\boldsymbol{v}\|_2 = 1} \|A\boldsymbol{v}\|_2, \tag{22}$$

where $\boldsymbol{\sigma}^\uparrow(A)$ is the vector of singular values in increasing order and the optimization is over subspaces of bounded dimension.

Total Points: 28 (+9)