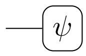
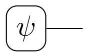
Exercise Sheet 4: Diagrams and Quantum Channels

Graphical calculus with tensor networks

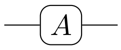
As you might have noticed, already for a little number of tensor factors even simple calculations can become hard to follow quite easily. Hence, an alternative approach to visualize such calculations was developed. We will give a short introduction into the basics of this calculation technique in this exercise. However, we encourage you to have a look into `https://arxiv.org/abs/1912.10049`, which gives a nice and complete overview over tensor networks. For this course, however, you won't need most of the content.

In tensor network notation, a tensor is simply an object that has indices, usually a set of complex numbers $A_{i_1,\ldots,i_n}$. A tensor with one index is a vector, a tensor with two indices is a matrix. A tensor with n indices is denoted as a box with n legs.

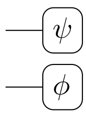
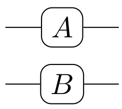
We have the following correspondences between the objects we already know and with diagrams. First, we will use the direction of the lines to distinguish kets and bras for states (which correspond to vectors):

 $\simeq |\psi\rangle$  $\simeq \langle\psi|$

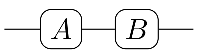
Linear operations, i.e. matrices, transform vectors into vectors and hence have one incoming leg and one outgoing leg:

 $\simeq \mathbb{I}$  $\simeq A$

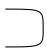
Tensor products can be expressed very easily by writing the same objects next to each other. This corresponds to the idea that tensor products represent all possible products between the entries of the objects, and as such we go from two objects with one index each to one object with two indices.

 $\simeq |\psi\rangle \otimes |\phi\rangle$  $\simeq A \otimes B$

One can think of each unconnected leg carrying a (dual) Hilbert space. Connecting two legs denotes contraction of the indices, so that for example the matrix product $[AB]_{ij} = \sum_{k=1}^{d} A_{ik}B_{kj}$ is denoted by

 $\simeq AB.$

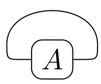
Another important primitive that we will use to reason in terms of diagrams is the fact that a bend of the wires is related to a maximally entangled state:

 $\simeq \sum_{i=1}^{d} |ii\rangle$

5 P. Exercise 1.

1 P. (a) Draw the inner product between two states $\langle\psi|\phi\rangle$ as a tensor product.

1 P. (b) Draw the expectation value $\langle\psi|A|\psi\rangle$ as a tensor network.

2 P. (c) What does the following tensor network represent? 

1 P. (d) Draw the expectation value for a mixed state, $\mathrm{Tr}[\rho A]$, as a tensor network.

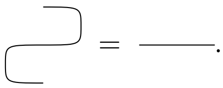
Before we can come to the next exercise, we have to clarify that in the context of tensor networks, we formally identify tensor products of kets and bras of computational basis states as outer products:

$$|i\rangle \otimes \langle j| \simeq |i\rangle\langle j|. \tag{1}$$

The following result is a very basic but important prerequisite for manipulating tensor diagrams called the snake equation which has already made it into a popular TV show you might have watched.

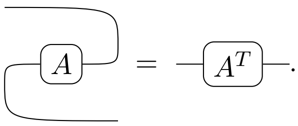
5 P. Exercise 2.

2 P. (a) Prove



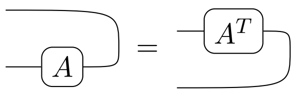
(2)

2 P. (b) Prove



(3)

1 P. (c) We emphasize that the results of the two preceding exercises also hold if you flip the tensor networks either horizontally or vertically. Use them to show that



(4)

Let us next come to an exercise that illustrates that concepts that are difficult to visualize through math are much more understandable when using tensor networks.

2 P. Exercise 3. A mixed quantum state ρ_{AB} on two systems can be written a tensor with four indices, two in two out:

$$\rho_{AB} = \sum_{i,j,k,l=1}^{d} \rho_{ijkl}(|i\rangle\langle j| \otimes |k\rangle\langle l|). \tag{5}$$

1 P. (a) Write the partial trace of ρ_{AB} over the system B as a tensor network diagram.

1 P. (b) Using tensor networks, prove the following statement from Exercise sheet 1

$$\mathrm{Tr}[\rho_{AB}(O_A \otimes \mathbb{I}_B)] = \mathrm{Tr}[\mathrm{Tr}_B[\rho_{AB}]O_A]. \tag{6}$$

As a bonus exercise, we give a proof of an identity that is surprisingly handy in quantum information theory and whose proof is very annoying when done by brute force, but very elegant using tensor networks.

4 P. Bonus Exercise 1. Let $\mathbb{F}$ denote the *flip operator* whose action on a tensor product is given by

$$\mathbb{F}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle. \tag{7}$$

2 P. (a) Draw the tensor network corresponding to $\mathbb{F}$.

2 P. (b) Prove that $\mathrm{Tr}[A^2] = \mathrm{Tr}[\mathbb{F}(A \otimes A)]$ using graphical calculus.

Quantum Channels

We have seen in the lecture as well as in previous exercise sheets that many of the notions in quantum information theory can be understood by starting with pure-state quantum mechanics and demanding a description for subsystems of such quantum systems. Some examples of this are the following statements

- Given an arbitrary pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ describing the joint state of two physical systems A and B, all measurement statistics of measurements on subsystem A (or B) are fully contained in the reduced density matrices $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ (or $\rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$). I.e. density matrices are required to describe the possible states of subsystems of larger systems whose states are pure.

- Given an arbitrary mixed state $\rho \in \mathcal{D}(\mathcal{H}_A)$ there always exists a second Hilbert space $\mathcal{H}_B$ and a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|$. (Such a $|\psi\rangle$ is called a *purification* of ρ). This means that all density matrices can be interpreted as states of a subsystem of a larger system which is in a pure state.

- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system (by Naimark's dilation theorem).

In this exercise we want to develop a similar picture for quantum channels by exploring the fact that quantum channels are exactly set of operations one can implement on a quantum system $\mathcal{H}_A$ by implementing a unitary operation on a joint system $\mathcal{H}_A \otimes \mathcal{H}_B$ and then looking at how the state of the subsystem A has transformed.

Kraus operators and Stinespring dilation

Recall that a linear map $\mathcal{N}\colon \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ is a proper quantum channel if and only if it is completely positive and trace preserving, which is equivalent to

$$\mathcal{N}\colon \rho \mapsto \sum_{k=1}^{l} E_k \rho E_k^\dagger \tag{8}$$

for some *Kraus operators* $\{E_k\}_{k=1}^l$ such that $\sum_{k=1}^l E_k^\dagger E_k = \mathbb{I}$.

In the following, we investigate the operational meaning of Kraus operators.

10 P. **Exercise 4.** For simplicity, we restrict ourselves to quantum channels with the same input and output space $\mathcal{N}\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. We now show that we can model such a quantum channel by adding an additional system with Hilbert space $\mathcal{Z}$ in state $|0\rangle\langle 0|$ and we apply a unitary U to the joint system and environment, i.e. we obtain a state

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger. \tag{9}$$

3 P. (a) Show that the action of any unitary on the joint system can be written as

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\langle l|,$$

with respect to the basis $\{|i\rangle\}_i$ on the second system $\mathcal{Z}$ for a set of some operators $\{E_k\}$. In particular show how these operators are related to the unitary U.

1 P. (b) Now, we perform a projective measurement on $\mathcal{Z}$ in the same basis. What is the probability of obtaining outcome i?

1 P. (c) Argue that the result of the previous exercise implies that

$$\sum_i E_i^\dagger E_i = \mathbb{I}. \tag{10}$$

1 P. (d) Determine the post-measurement state on the system $\mathcal{H}$ conditioned on the outcome i.

1 P. (e) Determine the state of the system $\mathcal{H}$ after the measurement on $\mathcal{Z}$ if the outcome of the measurement is unknown.

1 P. (f) Conclude that the procedure we just outlined implements a quantum channel – what are its Kraus operators?

2 P. (g) Use the results of the previous exercises to give an operational interpretation of the operators $\{E_i\}$.

As you learned in the lecture, the Kraus representation of a channel is usually not unique.

2 P. **Exercise 5.** Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{j=1}^N$ be two sets of Kraus operators. Show that if the two sets are related by a unitary transformation $U \in U(N)$ such that $\tilde{K}_j = \sum_i U_{ji} K_i$, the channels represented by the sets coincide.

Choi-Jamiołkowski isomorphism

In the lecture, you have already seen the Choi-Jamiołkowski isomorphism. For a quantum channel $\mathcal{N}$ acting on a d-dimensional Hilbert space $\mathcal{H}$, it is defined by the action of the channel on one half of a maximally entangled state on $\mathcal{H} \otimes \mathcal{H}$

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle, \quad \Omega = |\Omega\rangle\langle\Omega| \tag{11}$$

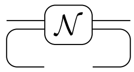
as

$$J_{\mathcal{N}} := (\mathbb{I} \otimes \mathcal{N})[\Omega]. \tag{12}$$

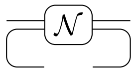
We also wish to emphasize that we could as well define $J_{\mathcal{N}}$ as $(\mathcal{N} \otimes \mathbb{I})[\Omega]$, we would still get a (different) isomorphism.

As a first exercise, we will use diagrammatic notation to elucidate that this is indeed an isomorphism.

3 P. **Exercise 6.** You can represent the the channel $\mathcal{N}$ as a diagram in the following way:

$$\mathcal{N} = \tag{13}$$


where the two open legs on the bottom correspond to the input of the channel where you would then put the input state to express $\mathcal{N}[\rho]$. Use this representation together with the representation of the maximally entangled state from the beginning of this sheet to establish that

$$J_{\mathcal{N}} \simeq \tag{14}$$


The Choi-Jamiołkowski isomorphism does more than enabling us to check the complete positivity of a quantum channel. We discuss one particular example below.

6 P. **Exercise 7.** Let $J_{\mathcal{N}}$ be the Choi state associated to a quantum channel $\mathcal{N}$. Then, we define the *unitarity* of a quantum channel as the purity of the Choi state:

$$U(\mathcal{N}) := \mathrm{Tr}[J_{\mathcal{N}}^2]. \tag{15}$$

This exercise is devoted to justifying this naming.

2 P. (a) Show that $U(\mathcal{N}) = 1$ if $\mathcal{N}[\rho] = V\rho V^\dagger$ for some unitary V.

2 P. (b) Show that $U(\mathcal{N}) = 1/d^2$ if the quantum channel maps any input state to the maximally mixed state $\mathcal{N}[\rho] = \mathrm{Tr}[\rho]\frac{\mathbb{I}}{d}$.

2 P. (c) Compute the unitarity of the channel $\mathcal{N}_p[\rho] = p\rho + \mathrm{Tr}[\rho](1-p)\frac{\mathbb{I}}{d}$ as a function of $p \in [0,1]$.

Recap

Both in the lecture and the exercise sheets we use mathematical and physical descriptions of quantum mechanics interchangeably. Here we revisit the way in which we talk about quantum states.

8 P. **Bonus Exercise 2.** Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a d_A- and d_B-dimensional Hilbert space, respectively. Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the tensor product space, with dimension $d_A \times d_B$. On the one hand, we want to recall the difference between a tensor product space and a Cartesian product space. On the other hand, we want to recall the difference between pure states and mixed states, which highlights some differences between the ket formulation of quantum mechanics (also called wave mechanics, championed by Schrödinger in the beginning) and the density operator formulation of quantum mechanics (also called matrix mechanics, championed by Heisenberg).

4 P. (a) Give the definitions of a pure state, a mixed state, a product state, and an entangled state. When possible, give the definitions both in terms of state kets[1] (unit-norm column vectors) and in terms of state density matrices.

2 P. (b) Let's now fix $\mathcal{H}_A = \mathcal{H}_B$ a 2-dimensional Hilbert space, and $\mathcal{H}$ correspondingly a 4-dimensional Hilbert space. Give a single example for a quantum state in each of the following categories, if possible, both in terms of kets and in terms of density matrices.

	Pure	Mixed
Product		
Entangled		

For product states, give both the expression as a single 2-qubit state, and as a product of two single-qubit states. For mixed states, also compute the purity explicitly.

[1] Careful, we want to be strict with the unit-norm part of the definition. A ket describes a quantum state only if its norm is equal to 1.

2 P. (c) Argue in what sense the Cartesian product space $\mathcal{H}_A \times \mathcal{H}_B$ can be thought of as being contained in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B \subseteq \mathcal{H}$. Which elements of $\mathcal{H}$ would not correspond to any elements in $\mathcal{H}_A \times \mathcal{H}_B$?

Total Points: 33 (+12)