Exercise Sheet 4: Diagrams and Quantum Channels

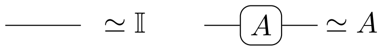
Graphical calculus with tensor networks

As you might have noticed, already for a little number of tensor factors even simple calculations can become hard to follow quite easily. Hence, an alternative approach to visualize such calculations was developed. We will give a short introduction into the basics of this calculation technique in this exercise. However, we encourage you to have a look into `https://arxiv.org/abs/1912.10049`, which gives a nice and complete overview over tensor networks. For this course, however, you won't need most of the content.

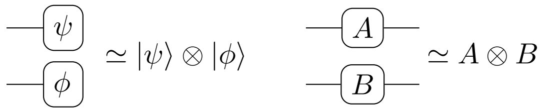
In tensor network notation, a tensor is simply an object that has indices, usually a set of complex numbers $A_{i_1,\ldots,i_n}$. A tensor with one index is a vector, a tensor with two indices is a matrix. A tensor with n indices is denoted as a box with n legs.

We have the following correspondences between the objects we already know and with diagrams. First, we will use the direction of the lines to distinguish kets and bras for states (which correspond to vectors):

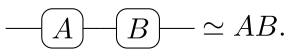
Linear operations, i.e. matrices, transform vectors into vectors and hence have one incoming leg and one outgoing leg:



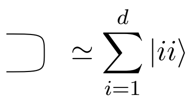
Tensor products can be expressed very easily by writing the same objects next to each other. This corresponds to the idea that tensor products represent all possible products between the entries of the objects, and as such we go from two objects with one index each to one object with two indices.



One can think of each unconnected leg carrying a (dual) Hilbert space. Connecting two legs denotes contraction of the indices, so that for example the matrix product $[AB]_{ij} = \sum_{k=1}^{d} A_{ik}B_{kj}$ is denoted by



Another important primitive that we will use to reason in terms of diagrams is the fact that a bend of the wires is related to a maximally entangled state:



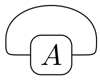
5 P. Exercise 1.

1 P. (a) Draw the inner product between two states $\langle\psi|\phi\rangle$ as a tensor product.

Solution

1 P. (b) Draw the expectation value $\langle\psi|A|\psi\rangle$ as a tensor network.

Solution

2 P. (c) What does the following tensor network represent? 

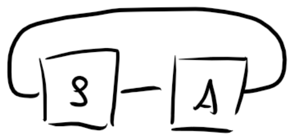
Solution

Connecting two open ends is contracting over the corresponding index, hence the tensor network represents

$$\sum_{i=1}^{d} A_{ii} = \mathrm{Tr}[A]. \tag{1}$$

1 P. (d) Draw the expectation value for a mixed state, $\mathrm{Tr}[\rho A]$, as a tensor network.

Solution



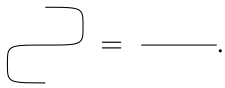
Before we can come to the next exercise, we have to clarify that in the context of tensor networks, we formally identify tensor products of kets and bras of computational basis states as outer products:

$$|i\rangle \otimes \langle j| \simeq |i\rangle\langle j|. \tag{2}$$

The following result is a very basic but important prerequisite for manipulating tensor diagrams called the snake equation which has already made it into a popular TV show you might have watched.

5 P. Exercise 2.

2 P. (a) Prove



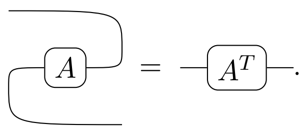
(3)

Solution

The tensor network represents the following analytical expression

$$
\begin{aligned}
\sum_{i,j=1}^{d} (\mathbb{I} \otimes \langle j| \otimes \langle j|)\,(|i\rangle \otimes |i\rangle \otimes \mathbb{I}) &= \sum_{i,j=1}^{d} |i\rangle \otimes \langle j|i\rangle \otimes \langle j| \\
&= \sum_{i=1}^{d} |i\rangle \otimes \langle i| \\
&\simeq \sum_{i=1}^{d} |i\rangle\langle i| \\
&= \mathbb{I}.
\end{aligned}
$$

2 P. (b) Prove



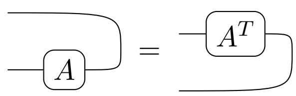
(4)

Solution

The proof proceeds in a similar way as in the previous exercise. The tensor network represents

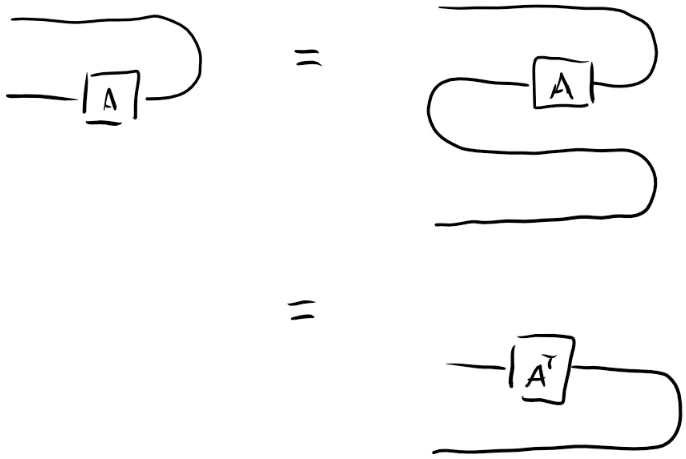
$$
\begin{aligned}
\sum_{i,j=1}^{d} (\mathbb{I} \otimes \langle j| \otimes \langle j|)\,(\mathbb{I} \otimes A \otimes \mathbb{I})\,(|i\rangle \otimes |i\rangle \otimes \mathbb{I}) &= \sum_{i,j=1}^{d} |i\rangle \otimes \langle j|A|i\rangle \otimes \langle j| \\
&= \sum_{i,j=1}^{d} A_{ji}\,(|i\rangle \otimes \langle i|) \\
&\simeq \sum_{i,j=1}^{d} A_{ji} |i\rangle\langle j| \\
&= \sum_{i,j=1}^{d} [A^T]_{ij} |i\rangle\langle j| \\
&= A^T.
\end{aligned}
$$

1 P. (c) We emphasize that the results of the two preceding exercises also hold if you flip the tensor networks either horizontally or vertically. Use them to show that



(5)

Solution



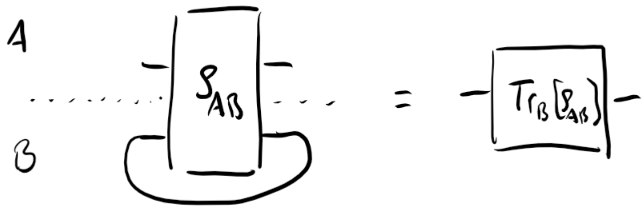
Let us next come to an exercise that illustrates that concepts that are difficult to visualize through math are much more understandable when using tensor networks.

2 P. **Exercise 3.** A mixed quantum state ρ_{AB} on two systems can be written a tensor with four indices, two in two out:

$$\rho_{AB} = \sum_{i,j,k,l=1}^{d} \rho_{ijkl}(|i\rangle\langle j| \otimes |k\rangle\langle l|). \tag{6}$$

1 P. (a) Write the partial trace of ρ_{AB} over the system B as a tensor network diagram.

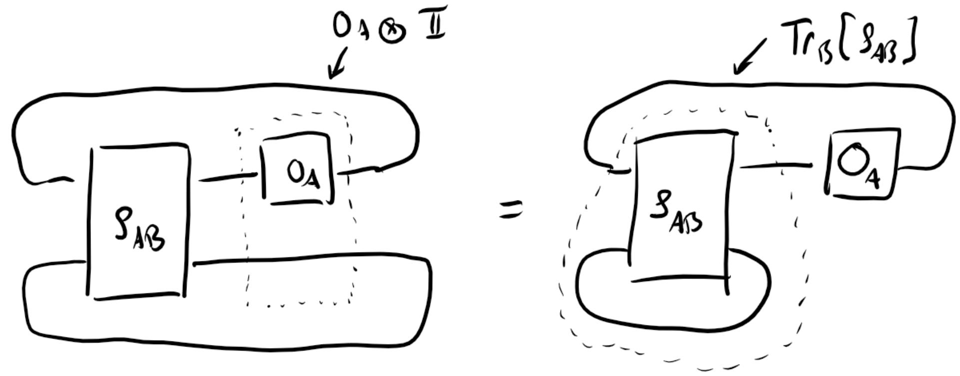
Solution



1 P. (b) Using tensor networks, prove the following statement from Exercise sheet 1

$$\mathrm{Tr}[\rho_{AB}(O_A \otimes \mathbb{I}_B)] = \mathrm{Tr}[\mathrm{Tr}_B[\rho_{AB}]O_A]. \tag{7}$$

Solution



As a bonus exercise, we give a proof of an identity that is surprisingly handy in quantum information theory and whose proof is very annoying when done by brute force, but very elegant using tensor networks.

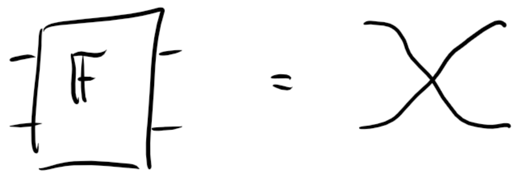
4 P. **Bonus Exercise 1.** Let $\mathbb{F}$ denote the *flip operator* whose action on a tensor product is given by

$$\mathbb{F}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle. \tag{8}$$

2 P. (a) Draw the tensor network corresponding to $\mathbb{F}$.

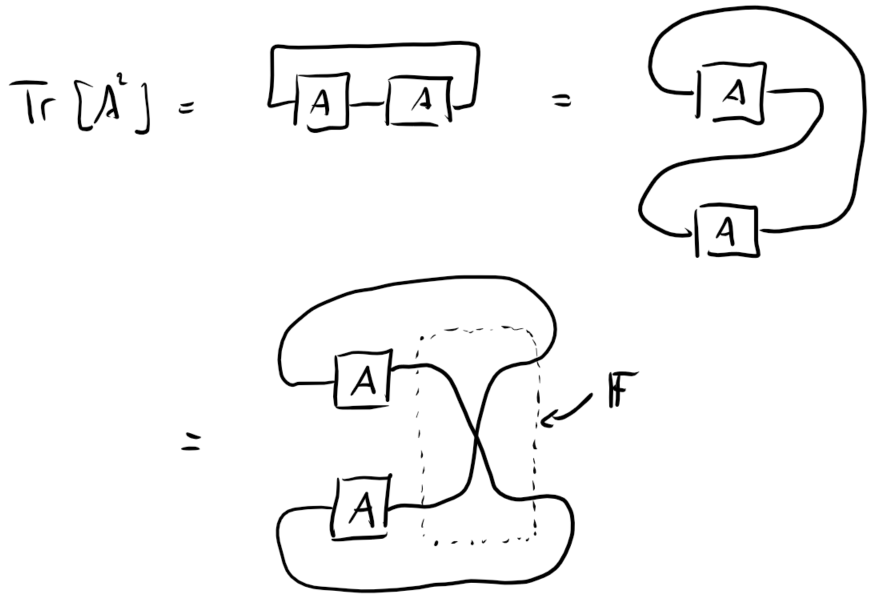
Solution

The action of $\mathbb{F}$ is to essentially exchange the indices of a tensor, we can hence write it as



2 P. (b) Prove that $\mathrm{Tr}[A^2] = \mathrm{Tr}[\mathbb{F}(A \otimes A)]$ using graphical calculus.

Solution



Quantum Channels

We have seen in the lecture as well as in previous exercise sheets that many of the notions in quantum information theory can be understood by starting with pure-state quantum mechanics and demanding a description for subsystems of such quantum systems. Some examples of this are the following statements

- Given an arbitrary pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ describing the joint state of two physical systems A and B, all measurement statistics of measurements on subsystem A (or B) are fully contained in the reduced density matrices $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ (or $\rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$). I.e. density matrices are required to describe the possible states of subsystems of larger systems whose states are pure.
- Given an arbitrary mixed state $\rho \in \mathcal{D}(\mathcal{H}_A)$ there always exists a second Hilbert space $\mathcal{H}_B$ and a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|$. (Such a $|\psi\rangle$ is called a *purification* of ρ). This means that all density matrices can be interpreted as states of a subsystem of a larger system which is in a pure state.
- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system (by Naimark's dilation theorem).

In this exercise we want to develop a similar picture for quantum channels by exploring the fact that quantum channels are exactly set of operations one can implement on a quantum system $\mathcal{H}_A$ by implementing a unitary operation on a joint system $\mathcal{H}_A \otimes \mathcal{H}_B$ and then looking at how the state of the subsystem A has transformed.

Kraus operators and Stinespring dilation

Recall that a linear map $\mathcal{N}\colon \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ is a proper quantum channel if and only if it is completely positive and trace preserving, which is equivalent to

$$\mathcal{N}\colon \rho \mapsto \sum_{k=1}^{l} E_k \rho E_k^\dagger \tag{9}$$

for some *Kraus operators* $\{E_k\}_{k=1}^l$ such that $\sum_{k=1}^l E_k^\dagger E_k = \mathbb{I}$.

In the following, we investigate the operational meaning of Kraus operators.

10 P. Exercise 4. For simplicity, we restrict ourselves to quantum channels with the same input and output space $\mathcal{N}\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. We now show that we can model such a quantum channel by adding an additional system with Hilbert space $\mathcal{Z}$ in state $|0\rangle\langle 0|$ and we apply a unitary U to the joint system and environment, i.e. we obtain a state

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger. \tag{10}$$

3 P. (a) Show that the action of any unitary on the joint system can be written as

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\langle l|\,,$$

with respect to the basis $\{|i\rangle\}_i$ on the second system $\mathcal{Z}$ for a set of some operators $\{E_k\}$. In particular show how these operators are related to the unitary U.

Solution

We define $E_k = (\mathbb{I} \otimes \langle k|)U(\mathbb{I} \otimes |0\rangle)$ and check:

$$\begin{aligned}
\sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\langle l| &= \sum_{kl} (\mathbb{I} \otimes \langle k|)U(\mathbb{I} \otimes |0\rangle)\rho\left((\mathbb{I} \otimes \langle l|)U(\mathbb{I} \otimes |0\rangle)\right)^\dagger \otimes |k\rangle\langle l| \\
&= \sum_{kl} (\mathbb{I} \otimes \langle k|)U(\mathbb{I} \otimes |0\rangle)\rho(\mathbb{I} \otimes \langle 0|)U^\dagger(\mathbb{I} \otimes |l\rangle) \otimes |k\rangle\langle l| \\
&= \sum_{kl} (\mathbb{I} \otimes \langle k|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(\mathbb{I} \otimes |l\rangle) \otimes |k\rangle\langle l| \\
&= U(\rho \otimes |0\rangle\langle 0|)U^\dagger\,.
\end{aligned}$$

If the last step isn't immediately clear to you, you can easily convince yourself by looking at matrix entries:

$$\begin{aligned}
&\langle i, m| \left(\sum_{kl} (\mathbb{I} \otimes \langle k|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(\mathbb{I} \otimes |l\rangle) \otimes |k\rangle\langle l| \right) |j, n\rangle \\
&= (\langle i| \otimes \mathbb{I}) \left(\sum_{kl} (\mathbb{I} \otimes \langle k|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(\mathbb{I} \otimes |l\rangle)\delta_{k,m}\delta_{l,n} \right) (|j\rangle \otimes \mathbb{I}) \\
&= (\langle i| \otimes \mathbb{I})(\mathbb{I} \otimes \langle m|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(\mathbb{I} \otimes |n\rangle)(|j\rangle \otimes \mathbb{I}) \\
&= \langle i, m|U(\rho \otimes |0\rangle\langle 0|)U^\dagger|j, n\rangle\,.
\end{aligned}$$

1 P. (b) Now, we perform a projective measurement on $\mathcal{Z}$ in the same basis. What is the probability of obtaining outcome i?

Solution

Using the solution of the previous exercise yields

$$\begin{aligned}
p(i) &= \mathrm{Tr}[(\mathbb{I} \otimes |i\rangle\langle i|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger] \\
&= \mathrm{Tr}[(\mathbb{I} \otimes |i\rangle\langle i|) \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\langle l|] \\
&= \mathrm{Tr}[E_i \rho E_i^\dagger] \\
&= \mathrm{Tr}[E_i^\dagger E_i \rho].
\end{aligned}$$

1 P. (c) Argue that the result of the previous exercise implies that

$$\sum_i E_i^\dagger E_i = \mathbb{I}. \tag{11}$$

Solution

We have that

$$1 = \sum_i p(i) = \sum_i \mathrm{Tr}[E_i^\dagger E_i \rho].$$

As this holds for any ρ, we necessarily have that

$$\sum_i E_i^\dagger E_i = \mathbb{I}.$$

1 P. (d) Determine the post-measurement state on the system $\mathcal{H}$ conditioned on the outcome i.

Solution

The state on the system is

$$\frac{E_i \rho E_i^\dagger}{\mathrm{Tr}[E_i \rho E_i^\dagger]} = \frac{E_i \rho E_i^\dagger}{p(i)}.$$

If not tracing out the additional environment $\mathcal{Z}$, the post measurement state is

$$\frac{1}{p(i)} E_i \rho E_i^\dagger \otimes |i\rangle\langle i|.$$

1 P. (e) Determine the state of the system $\mathcal{H}$ after the measurement on $\mathcal{Z}$ if the outcome of the measurement is unknown.

Solution

Using the result of the previous exercise, we have

$$\sum_i p(i) \frac{E_i \rho E_i^\dagger}{p(i)} = \sum_i E_i \rho E_i^\dagger.$$

If not tracing out the additional environment $\mathcal{Z}$, the post measurement state is

$$\sum_i p(i) \frac{E_i \rho E_i^\dagger}{p(i)} \otimes |i\rangle\langle i| = \sum_i E_i \rho E_i^\dagger \otimes |i\rangle\langle i|.$$

1 P. (f) Conclude that the procedure we just outlined implements a quantum channel – what are its Kraus operators?

Solution

This is a quantum channel with Kraus operators $\{E_i\}$. The fact that they form proper Kraus operators was established in Ex. (c).

2 P. (g) Use the results of the previous exercises to give an operational interpretation of the operators $\{E_i\}$.

Solution

The $E_i^\dagger E_i$ can be seen as elements of a POVM implemented on the first system by the von-Neumann (projective) measurement on the second system.

As you learned in the lecture, the Kraus representation of a channel is usually not unique.

2 P. **Exercise 5.** Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{j=1}^N$ be two sets of Kraus operators. Show that if the two sets are related by a unitary transformation $U \in U(N)$ such that $\tilde{K}_j = \sum_i U_{ji} K_i$, the channels represented by the sets coincide.

Solution

We have

$$\begin{aligned}\tilde{\mathcal{N}}(X) &= \sum_j \tilde{K}_j X \tilde{K}_j^\dagger = \sum_{ijk} U_{ji} K_i X (U_{jk} K_k)^\dagger \\ &= \sum_{ik} \left(\sum_j U_{ji} U_{jk}^* \right) K_i X K_k^\dagger = \sum_{ik} \delta_{ki} K_i X K_k^\dagger = \mathcal{N}(X),\end{aligned}$$

where in the last equality we used the unitarity of U, which tells us that $\sum_j U_{ji} U_{jk}^* = \sum_j (U^\dagger)_{kj} U_{ji} = (U^\dagger U)_{ki} = \delta_{ki}$.

Choi-Jamiołkowski isomorphism

In the lecture, you have already seen the Choi-Jamiołkowski isomorphism. For a quantum channel $\mathcal{N}$ acting on a d-dimensional Hilbert space $\mathcal{H}$, it is defined by the action of the channel on one half of a maximally entangled state on $\mathcal{H} \otimes \mathcal{H}$

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle, \quad \Omega = |\Omega\rangle\langle\Omega| \tag{12}$$

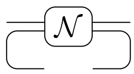
as

$$J_{\mathcal{N}} := (\mathbb{I} \otimes \mathcal{N})[\Omega]. \tag{13}$$

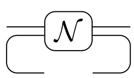
We also wish to emphasize that we could as well define $J_{\mathcal{N}}$ as $(\mathcal{N} \otimes \mathbb{I})[\Omega]$, we would still get a (different) isomorphism.

As a first exercise, we will use diagrammatic notation to elucidate that this is indeed an isomorphism.

3 P. **Exercise 6.** You can represent the the channel $\mathcal{N}$ as a diagram in the following way:

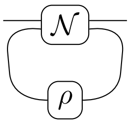
$\mathcal{N} =$  (14)

where the two open legs on the bottom correspond to the input of the channel where you would then put the input state to express $\mathcal{N}[\rho]$. Use this representation together with the representation of the maximally entangled state from the beginning of this sheet to establish that

$J_{\mathcal{N}} \simeq$  (15)

Solution

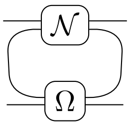
A CPTP map $\mathcal{N}$ acting on a density matrix ρ can be represented as

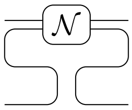
$\mathcal{N}[\rho] =$ 

The density matrix of the maximally entangled state, Ω, can be represented as

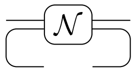
$\Omega = \frac{1}{d}$ 

Now we can draw $J_{\mathcal{N}}$ as

$J_{\mathcal{N}} = (\mathcal{N} \otimes \mathbb{I})(\Omega) =$ 

$= \frac{1}{d}$ 

$= \frac{1}{d}$ 

$\simeq \frac{1}{d}$ 

The isomorphism in the last step is intuitive, as bending a wire is an isomorphism of tensor networks.

The Choi-Jamiołkowski isomorphism does more than enabling us to check the complete positivity of a quantum channel. We discuss one particular example below.

6 P. **Exercise 7.** Let $J_{\mathcal{N}}$ be the Choi state associated to a quantum channel $\mathcal{N}$. Then, we define the *unitarity* of a quantum channel as the purity of the Choi state:

$$U(\mathcal{N}) := \mathrm{Tr}[J_{\mathcal{N}}^2]. \tag{16}$$

This exercise is devoted to justifying this naming.

2 P. (a) Show that $U(\mathcal{N}) = 1$ if $\mathcal{N}[\rho] = V\rho V^\dagger$ for some unitary V.

Solution

If $\mathcal{N}$ is unitary, then

$$J_{\mathcal{N}} = (\mathbb{I} \otimes V)|\Omega\rangle\langle\Omega|(\mathbb{I} \otimes V^\dagger), \tag{17}$$

which means the Choi state is a pure state undergoing unitary evolution. Hence, the Choi state is a pure quantum state and therefore has $\mathrm{Tr}[J_{\mathcal{N}}^2] = 1$.

2 P. (b) Show that $U(\mathcal{N}) = 1/d^2$ if the quantum channel maps any input state to the maximally mixed state $\mathcal{N}[\rho] = \mathrm{Tr}[\rho]\frac{\mathbb{I}}{d}$.

Solution

The depolarizing channel acting on one half of a maximally entangled state means that we trace out that half of the maximally entangled state and replace it with a maximally mixed state. Because the state was maximally entangled, the other half of the state is also in a maximally mixed state, which means that

$$J_{\mathcal{N}} = \frac{\mathbb{I}}{d^2}$$

which has purity $\mathrm{Tr}[J_{\mathcal{N}}^2] = \frac{1}{d^2}$.

2 P. (c) Compute the unitarity of the channel $\mathcal{N}_p[\rho] = p\rho + \mathrm{Tr}[\rho](1-p)\frac{\mathbb{I}}{d}$ as a function of $p \in [0, 1]$.

Solution

Combining the results of the previous exercises means that the Choi state of this channel is

$$J_{\mathcal{N}_p} = p|\Omega\rangle\langle\Omega| + (1-p)\frac{\mathbb{I}}{d^2}.$$

We can compute its purity directly

$$\begin{aligned}
\mathrm{Tr}[J^2_{\mathcal{N}_p}] &= \mathrm{Tr}\left[\left(p|\Omega\rangle\langle\Omega| + (1-p)\frac{\mathbb{I}}{d^2}\right)^2\right] \\
&= \mathrm{Tr}\left[p^2|\Omega\rangle\langle\Omega| + 2p(1-p)\frac{|\Omega\rangle\langle\Omega|}{d^2} + (1-p)^2\frac{\mathbb{I}}{d^4}\right] \\
&= p^2 + \frac{2p(1-p)}{d^2} + \frac{(1-p)^2}{d^2} \\
&= p^2 + (1-p^2)\frac{1}{d^2}.
\end{aligned}$$

Recap

Both in the lecture and the exercise sheets we use mathematical and physical descriptions of quantum mechanics interchangeably. Here we revisit the way in which we talk about quantum states.

8 P. **Bonus Exercise 2.** Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a d_A- and d_B-dimensional Hilbert space, respectively. Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the tensor product space, with dimension $d_A \times d_B$. On the one hand, we want to recall the difference between a tensor product space and a Cartesian product space. On the other hand, we want to recall the difference between pure states and mixed states, which highlights some differences between the ket formulation of quantum mechanics (also called wave mechanics, championed by Schrödinger in the beginning) and the density operator formulation of quantum mechanics (also called matrix mechanics, championed by Heisenberg).

4 P. (a) Give the definitions of a pure state, a mixed state, a product state, and an entangled state. When possible, give the definitions both in terms of state kets[1] (unit-norm column vectors) and in terms of state density matrices.

[1] Careful, we want to be strict with the unit-norm part of the definition. A ket describes a quantum state only if its norm is equal to 1.

Solution

Product states are easy to define: they are the elements $|\omega\rangle \in \mathcal{H}$ for which there exist $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$ such that $|\omega\rangle = |\psi\rangle \otimes |\phi\rangle$. Then, the non-product states are those for which the above is not true. A pure state is called entangled if and only if it is non-product. For mixed states it's not so direct. A convex combination of product states $\rho = \sum_i p(i)\rho_{1,i} \otimes \rho_{2,i}$, for $\sum_i p(i) = 1$ is not necessarily product, yet its "randomness" comes only from the classical mixing, so it's not the same as entanglement. Convex combinations of product states are called *separable states.* A mixed state is called entangled if and only if it is not separable, aka, if and only if it cannot be written as a convex combination of tensor products.
A pure state is a state that can be expressed as a single ket. For the density matrix formulation, a pure state is a rank-one projector. A mixed state can be understood as a classical mixture of pure states, or as the reduced state of a larger system with entanglement. We would say a mixed state is then any density matrix whose rank is larger than one. General mixed states cannot be written down as single kets, let us elaborate briefly, having the classical mixture picture in mind:

(a) Let's say we have a classical mixture, which means the physical system could be in a number of different states $\{|\psi_i\rangle_i\}$, each with a certain probability $p(i)$. We want to find a mathematical description of the physical state including the classical probability distribution. We might be tempted to write $\sum_i p(i)|\psi_i\rangle$, but in general this is not a valid quantum state, because the resulting vector does not have norm one in general. We might be tempted to then just renormalize the resulting vector, which would give rise to the state $|\psi\rangle = \frac{1}{\sum_j p(j)^2} \sum_i p(i)|\psi_i\rangle$. But this corresponds to a quantum superposition, not to a classical mixture. This highlights the difference between the statements "this state is in a superposition of $|0\rangle$ and $|1\rangle$" and "this state could be either $|0\rangle$ or $|1\rangle$ with a certain probability". The difference might seem irrelevant at first in that, if we were to measure the state, the same outcome statistics would arise from both situations. Nevertheless, if we were to apply any quantum transformation on the state before measuring, then we would observe different behaviors.

(b) Let's stay with the same classical mixture, just now we take the density matrix formulation of the states $\{\rho_i\}_i$. We still want to find the mathematical description of the physical state including the classical probability distribution. We again are tempted to write $\sum_i p(i)\rho_i$. And, this time, we notice we're left with a valid density matrix! Indeed, $\rho = \sum_i p(i)\rho_i$ is a Hermitian, PSD, trace-one matrix.

What is then the difference between both pictures? Could we not just find a different ket that does the same job as the density matrix? W.l.o.g. let's assume we take d quantum states which are pairwise orthogonal: $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. Then, consider the density matrix $\rho = \sum_i p(i)\rho_i$ on the one hand, and a hypothetical ket $|\Psi\rangle$ that should correspond to the same thing on the other hand. We can immediately observe that the density matrix corresponding to this new ket $|\Psi\rangle\langle\Psi|$ is a rank-one matrix, irrespective of the choice of ket. At the same time, we know by construction that ρ is rank d. It follows that there is no description based just on kets which can capture the situation of a classical mixture, and in this way matrix mechanics is more general than wave mechanics.

2 P. (b) Let's now fix $\mathcal{H}_A = \mathcal{H}_B$ a 2-dimensional Hilbert space, and $\mathcal{H}$ correspondingly a 4-dimensional Hilbert space. Give a single example for a quantum state in each of the following categories, if possible, both in terms of kets and in terms of density matrices.

	Pure	Mixed
Product		
Entangled		

For product states, give both the expression as a single 2-qubit state, and as a product of two single-qubit states. For mixed states, also compute the purity explicitly.

Solution

A pure, product state: $|00\rangle$, $\mathrm{diag}(1,0,0,0)$. Which is the same as $|0\rangle \otimes |0\rangle$, $\mathrm{diag}(1,0) \otimes \mathrm{diag}(1,0)$.
A pure, entangled state: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

A mixed, product state: $\frac{1}{4}\mathbb{I}_4 = (\frac{1}{2}\mathbb{I}_2) \otimes (\frac{1}{2}\mathbb{I}_2)$, cannot be expressed as a single ket. Purity is $\frac{1}{16}\operatorname{Tr}[\mathbb{I}_4] = \frac{1}{4}$, it's the maximally mixed state.
A mixed, entangled state: $p|\Phi^+\rangle\langle\Phi^+| + (1-p)|\Phi^-\rangle\langle\Phi^-|$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 2p-1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2p-1 & 0 & 0 & 1 \end{pmatrix}, \tag{19}$$

cannot be expressed as a single ket. Purity is $p^2 + (1-p)^2$.

2 P. (c) Argue in what sense the Cartesian product space $\mathcal{H}_A \times \mathcal{H}_B$ can be thought of as being contained in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B \subseteq \mathcal{H}$. Which elements of $\mathcal{H}$ would not correspond to any elements in $\mathcal{H}_A \times \mathcal{H}_B$?

Solution

The Cartesian product space can be identified with the subspace of product states in the tensor product space. In this way, for any $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$, we can define an inclusion map $|\psi\rangle \times |\phi\rangle \hookrightarrow |\psi\rangle \otimes |\phi\rangle$ to loosely argue that $\mathcal{H}_A \times \mathcal{H}_B \subseteq \mathcal{H}$. This is not a formal mathematical statement, although it could be turned into one if we really wanted to (by arguing that there is a bijection between the Cartesian product space and the subspace of product states in the tensor product space).
Under the above defined inclusion map (which would be the standard one), the elements of $\mathcal{H}$ that do not correspond to any elements in $\mathcal{H}_A \times \mathcal{H}_B$ are the "non-product" elements. Product states are easy to define: they are the elements $|\omega\rangle \in \mathcal{H}$ for which there exist $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$ such that $|\omega\rangle = |\psi\rangle \otimes |\phi\rangle$. Then, the non-product states are those for which the above is not true. An example of such a non-product state would be, given $\{|a_o\rangle, |a_1\rangle\} \subseteq \mathcal{H}_A$ and $\{|b_0\rangle, |b_1\rangle\} \subseteq \mathcal{H}_B$ two pairs of orthogonal states, $\frac{1}{\sqrt{2}}(|a_0 b_0\rangle + |a_1 b_1\rangle) \in \mathcal{H} \setminus (\mathcal{H}_A \times \mathcal{H}_B)$.

Total Points: 33 (+12)