Exercise Sheet 5: Entropies and Examples of Quantum Channels

Entropic Quantities

Properties of the Shannon entropy

The Shannon entropy and related entropic quantities are of fundamental importance in classical information theory. We now invest some time and effort to better understand these notions.

8 P. **Exercise 1.** Recall the definition of the Shannon entropies for random variables X, Y which take values in (for simplicity finite) sets $\mathcal{X}, \mathcal{Y}$, and are distributed according to probability distributions p, q over $\mathcal{X}$ and $\mathcal{Y}$, respectively:

$$\text{Shannon entropy:} \qquad H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x) \tag{1}$$

$$\text{Conditional entropy:} \qquad H(X|Y) = H(X,Y) - H(Y) \qquad = \sum_{y\in\mathcal{Y}} p(y) H(X|Y=y) \tag{2}$$

$$= -\sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(y)}$$

Note that here, as is common in information theory, log denotes the logarithm with base 2. For these definitions, recall that we take the convention $0\log(0) \equiv 0$.

2 P. (a) Show that $0 \leq H(X) \leq \log|\mathcal{X}|$. Moreover, show that the first equality holds if there is an $x \in \mathcal{X}$ for which $p(x) = 1$, and that the second equality holds if $p(x) = 1/|\mathcal{X}|$ for all x.

Hint: To prove the second inequality, use Jensen's inequality for a suitably chosen concave function.

3 P. (b) Show that the Shannon entropy is *subadditive*, i.e., that $H(X,Y) \leq H(X) + H(Y)$ with equality if X and Y are independent.

Hint: Show that $H(X,Y) - H(X) - H(Y) \leq 0$ *using that* $\ln(2)\cdot\log_2(x) = \ln(x) \leq x - 1$.

2 P. (c) Show that $H(Y|X) \geq 0$, with equality if Y is a (deterministic) function of X.

Hint: Use Bayes' rule: $p(x,y) = p(y|x)p(x)$.

1 P. (d) Show that $H(Y|X) \leq H(Y)$ with equality if X and Y are independent random variables.

Properties of the von Neumann entropy

Above, we have investigated some useful properties of the Shannon entropy. Now, we do the same for the von Neumann entropy. For any state $\rho \in \mathcal{D}(\mathcal{H})$ with $\dim\mathcal{H} = d$, the von Neumann entropy is defined as $S(\rho) = -\operatorname{Tr}[\rho\log\rho]$. Recall from previous exercise sheets that $\log\rho$ is obtained by taking the eigendecomposition $\rho = UDU^\dagger$, with $D = \operatorname{diag}((\lambda_i)_i)$ and applying the log function to the (non-zero) eigenvalues, leaving everything else untouched $\log\rho = U\operatorname{diag}((\log\lambda_i)_i)V^\dagger$.

6 P. **Exercise 2.**

1 P. (a) Show that $0 \leq S(\rho) \leq \log d$ with equality if and only if ρ is pure.

Hint: You've already proved something quite similar for the Shannon entropy in Exercise 1. So, don't start from scratch but instead try to use that already established result.

2 P. (b) Prove that for any two positive definite matrices A and B we have $\log(A \otimes B) = \log(A) \otimes \mathbb{I} + \mathbb{I} \otimes \log(B)$.

Note: We only ask you to prove for $A, B > 0$, but this result can be extended to $A, B \geq 0$ given a proper definition of the matrix logarithm.

3 P. (c) Show that the von Neumann entropy is *subadditive* in the following sense: If two distinct systems A and B have a joint quantum state ρ_{AB} then $S(A, B) \leq S(A) + S(B)$, with equality if $\rho_{AB} = \rho_A \otimes \rho_B$.

Hint: You may use the inequality $S(\rho) \leq -\operatorname{Tr}[\rho \log \sigma]$ for an arbitrary quantum state σ (without proving it).

4 P. Bonus Exercise 1.

3 P. (a) Suppose that $p = (p_i)_i$ is a discrete probability distribution and that the states ρ_i are mutually orthogonal. Show that

$$S\left(\sum_i p_i \rho_i\right) = H(p) + \sum_i p_i S(\rho_i).$$

and use this result to infer that

$$S\left(\sum_i p_i \rho_i \otimes |\psi_i\rangle\langle\psi_i|\right) = H(p) + \sum_i p_i S(\rho_i),$$

where $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ and the ρ_i are arbitrary quantum states.

1 P. (b) Use the subadditivity and the result from (a) to infer that the von Neumann entropy S is concave, that is, $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$ for any probability distribution $\{p_i\}$ and states ρ_i.

5 P. Exercise 3. Now we want to set our sight back on bipartite systems and entanglement entropies. Throughout this problem, if the global state being referred to is clear, we will denote entropies of the reduced states using the corresponding Hilbert space as an argument, e.g. the entropy of a state ρ_{AB} reduced on subsystem A, that is $S\left(\operatorname{Tr}_B\left[\rho_{AB}\right]\right)$, is denoted $S(A)$.

1 P. (a) Let $|\psi\rangle_{AB} = |\xi\rangle_A \otimes |\eta\rangle_B$ be some product pure state. Compute the entanglement entropy of the state, as well as the conditional von Neumann entropy $S(A|B) \equiv S(A, B) - S(B)$.

1 P. (b) Let Ω_{AB} be the maximally entangled state on two Hilbert spaces of equal dimension d, i.e. $\Omega = |\Omega\rangle\langle\Omega|$ with

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle.$$

Compute the conditional von Neumann entropy $S(A|B) \equiv S(A, B) - S(B)$. What do you conclude when comparing your result to that of Exercise 1(c)?

2 P. (c) Let ρ_{AB} be a bipartite (potentially mixed) state. Show that if ρ_A is separable, i.e. $\rho_{AB} = \sum_i p_i \sigma_A^i \otimes \tau_B^i$ where σ^i and τ^i are states and $\{p_i\}$ is a probability distribution, then $S(A|B) \geq 0$.

Hint: The concavity of the von Neumann entropy established in (e) can be helpful here.

1 P. (d) In the lecture, you have encountered the entropy of entanglement, defined via the entropy of a reduced density matrix, as a measure for pure state entanglement. Show via an example that the entropy of a reduced density matrix is not suitable for quantifying entanglement in mixed states.

Examples of Quantum Channels

On the last sheet, we looked at quantum channels on a general, abstract level. Here, we investigate some examples of quantum channels acting on qubits, i.e., our Hilbert space is $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$\begin{aligned} F_\epsilon(\rho) &:= \epsilon X \rho X + (1-\epsilon)\rho) \\ D_\epsilon(\rho) &:= \epsilon \operatorname{Tr}[\rho]\frac{\mathbb{I}}{2} + (1-\epsilon)\rho \\ A_\epsilon(\rho) &:= \epsilon \operatorname{Tr}[\rho]|0\rangle\langle 0| + (1-\epsilon)\rho, \end{aligned}$$

where $\epsilon \in [0,1]$.

10 P. **Exercise 4.** In this exercise, we will familiarize ourselves with the three channels above. In the process, we will also learn to use the three important representations of channels that we have encountered so far – Choi-Jamiołkowski, Kraus, and Stinespring – in concrete examples rather than just abstractly.

4 P. (a) Show that F_ϵ, D_ϵ, and A_ϵ indeed are linear CPTP maps and thus valid quantum channels.

3 P. (b) Describe the action of each of the channels F_ϵ, D_ϵ, and A_ϵ in words.

3 P. (c) Compute the action of each of the three channels on the input states $|0\rangle\langle 0|$ and $\rho = \mathbb{I}/2$.

Recap

As we keep gaining in speed and confidence when talking about mixed and entangled states, we take a moment to revisit some core concepts via direct, small examples.

5 P. **Bonus Exercise 2.**

3 P. (a) Come up with three mixed quantum states as density matrices ρ_1, ρ_2, and ρ_3. Let the states be such that the density matrix is not diagonal in the computational basis, and let each of them have the corresponding purity: $\operatorname{Tr}(\rho_k^2) = k/(k+1)$.

2 P. (b) Compute the purity of the product states $\rho_{i,j} = \rho_i \otimes \rho_j$ for every combination of $i, j \in \{1,2,3\}$.

Hint: There is of course a brute force solution, but spend a few minutes thinking about whether you can save yourself computational effort.

3 P. **Bonus Exercise 3.** Consider a mixed state as a classical mixture of pure states.

1 P. (a) Can you obtain an entangled mixed state by mixing only pure tensor product states? (Either give an illustrative example or prove that it's not possible.)

1 P. (b) Can you obtain a tensor product mixed state by mixing only pure entangled states? (Either give an illustrative example or prove that it's not possible.)

1 P. (c) Given the results of (a) and (b), what do you conclude about how entanglement behaves under probabilistic mixtures?

Total Points: 29 (+12)