Exercise Sheet 6: LOCC and Majorization

LOCC

At the heart of entanglement theory lies the notion of LOCC (Local Operations and Classical Communication). To see why, imagine two parties that are a large distance apart from each other, say, Alice is in Berlin and Bob in New York. While they may obtain access to shared entanglement from a third party, it is unreasonable to assume that they are able to perform global operations on the state they share. However, it is perfectly conceivable that they transmit classical messages, for example, to communicate measurement results.

The goal of the next exercise is to show that if Alice and Bob are in distant labs, and share a state, any measurement on Alice's part of the state can be simulated as follows: Bob performs a measurement on his side and communicates the result to Alice, who performs a local unitary transformation. This can be proven for POVMs, but for simplicity we will restrict ourselves to projective measurements.

8 P. **Exercise 1.** We work on a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. For convenience we assume $\mathcal{H}_A \cong \mathcal{H}_B$. Consider a bipartite pure state $|\psi\rangle_{AB}$ with Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |a_i\rangle |b_i\rangle \tag{1}$$

and a projective measurement $\Pi = \{\Pi_i^A\}_i$ acting on Alice's Hilbert space.

3 P. (a) Expand Π_i^A in the basis of the Schmidt decomposition and define a projective measurement $\Gamma = \{\Gamma_i^B\}_i$ on Bob's system such that the probability p_k^B that Bob observes result k when measuring Γ is the same as the probability p_k^A that Alice observes result k when measuring Π.

2 P. (b) Determine the post-measurement states $|\phi_j^A\rangle$ after Alice measures Π and obtains result j, and $|\phi_j^B\rangle$ after Bob measures Γ and obtains result j. (Note: Both of these states are defined on the whole Hilbert space AB, the superscripts serve to identify who performed the measurement).

2 P. (c) Show that $|\phi_j^A\rangle$ and $|\phi_j^B\rangle$ are equivalent up to local unitary transformations.

Hint: Recall that for a state of the form $\sum_{k\ell} C_{k\ell}|e_k\rangle|f_\ell\rangle$, with orthonormal sets $\{|e_k\rangle\}_k$ and $\{|f_\ell\rangle\}_\ell$, its Schmidt coefficients are exactly the singular values of the matrix $C = (C_{k\ell})_{k\ell}$.

1 P. (d) Describe the LOCC protocol.

Majorization

Majorization is a mathematical concept that has surprisingly far-reaching applications. Consider two vectors $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n$. We define a sorted (in descending manner) version of a vector $\boldsymbol{v}$ as $\boldsymbol{v}^\downarrow$, such that

$$v_1^\downarrow \geq v_2^\downarrow \geq \ldots \geq v_n^\downarrow. \tag{2}$$

We can write

$$\boldsymbol{v}^\downarrow = P\boldsymbol{v} \tag{3}$$

for some matrix P that permutes the entries of $\boldsymbol{v}$.

2 P. **Exercise 2.** Let us look at the vector $\boldsymbol{y} = (2,4,1,3)$. What is $\boldsymbol{y}^{\downarrow}$ and what matrix P permutes $\boldsymbol{y}$ into $\boldsymbol{y}^{\downarrow}$?

We now say that $\boldsymbol{x}$ *majorizes* $\boldsymbol{y}$, written as $\boldsymbol{x} \succ \boldsymbol{y}$, if

$$\boldsymbol{x} \succ \boldsymbol{y} \;\Leftrightarrow\; \sum_{j=1}^{k} x_j^{\downarrow} \geq \sum_{j=1}^{k} y_j^{\downarrow} \text{ for all } 1 \leq k \leq n. \tag{4}$$

1 P. **Exercise 3.** Show that $\boldsymbol{x} = (2,1,0)$ majorizes $\boldsymbol{y} = (1,1,1)$.

A central insight in the theory of majorization is that the majorization condition $\boldsymbol{x} \succ \boldsymbol{y}$ holds if and only if we can write

$$\boldsymbol{x} \succ \boldsymbol{y} \;\Leftrightarrow\; \boldsymbol{y} = \sum_j p_j P_j \boldsymbol{x} \tag{5}$$

for a probability distribution p_j over permutation matrices P_j. Birkhoff's theorem then implies that we can write

$$\boldsymbol{y} = D\boldsymbol{x} \tag{6}$$

for a *doubly-stochastic* matrix D, which is a matrix where all columns and rows are simultaneously probability distributions, i.e.

$$D \text{ is doubly-stochastic } \;\Leftrightarrow\; D_{ij} \geq 0 \text{ and } \sum_{i=1}^{n} D_{ij} = 1 \text{ and } \sum_{j=1}^{n} D_{ij} = 1 \text{ for all } i \text{ and } j. \tag{7}$$

The purpose of the next exercise is to uncover the role of majorization in state transformations using LOCC. Specifically, we will try to understand when we can transform a given copy of a pure bipartite quantum state $|\psi\rangle$ to another quantum state $|\phi\rangle$ using LOCC, which we write as

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle. \tag{8}$$

As a first ingredient, we extend the definition of majorization to Hermitian matrices. Let X, Y be Hermitian $n \times n$ matrices. If $\boldsymbol{\lambda}(X)$ ($\boldsymbol{\lambda}(Y)$) is the vector of eigenvalues of X (Y), then we say that X majorizes Y if their vectors of eigenvalues majorize each other, i.e.

$$X \succ Y \;\Leftrightarrow\; \boldsymbol{\lambda}(X) \succ \boldsymbol{\lambda}(Y). \tag{9}$$

4 P. **Exercise 4.** Show that $X \succ Y$ for Hermitian X and Y if and only if there exists a probability distribution p and unitary matrices U_j such that

$$Y = \sum_j p_j U_j X U_j^{\dagger}.$$

Hint: Make use of the eigendecompositions of X and Y and use Eq. (5) *for the "if" part and Eq.* (6) *for the "only if" part.*

6 P. **Exercise 5.** We are now ready to prove the surprising theorem that conversion of $|\psi\rangle$ into $|\phi\rangle$ under LOCC is only possible if the reduced state $\mathrm{Tr}_B[|\phi\rangle\langle\phi|]$ majorizes the reduced state $\mathrm{Tr}_B[|\psi\rangle\langle\psi|]$:

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \;\Leftrightarrow\; \mathrm{Tr}_B[|\psi\rangle\langle\psi|] \prec \mathrm{Tr}_B[|\phi\rangle\langle\phi|]. \tag{10}$$

We encourage you to have a look into the original paper that established the theorem (`https://arxiv.org/pdf/quant-ph/9811053.pdf`).

3 P. (a) Show the “only if” direction in Eq. (10) using the previous result, i.e. assume that $|\psi\rangle$ can be transformed into $|\phi\rangle$ under LOCC and show that this implies the states majorization condition. You can suppose that LOCC is realised by a measurement $M = \{M_j\}$ on Alice's side and a corresponding unitary on Bob's side. In other words, from Alice's point of view it must be the case that[1]

$$M_j \mathrm{Tr}_B[|\psi\rangle\langle\psi|] M_j^\dagger = p_j \mathrm{Tr}_B[|\phi\rangle\langle\phi|]. \tag{11}$$

Hint: Define $X_j := M_j \sqrt{\mathrm{Tr}_B[|\psi\rangle\langle\psi|]}$ such that the left hand side of the above equations is $X_j X_j^\dagger$ and use the polar decomposition of X and the fact that $\{M_j\}$ is a POVM.

3 P. (b) Now show the “if” direction in Eq. (10) by proceeding analogously.

5 P. Bonus Exercise 1. Let's try to apply Eq. (10) a bit more concretely.

3 P. (a) Give one example for each of the following:

(i) A pair of pure 2-qubit states $(|\psi\rangle, |\phi\rangle)$ such that $|\psi\rangle$ can be LOCC-transformed into $|\phi\rangle$ but not vice versa.

(ii) A pair of pure 2-qubit states $(|\psi\rangle, |\phi\rangle)$ such that $|\psi\rangle$ can be LOCC-transformed into $|\phi\rangle$ and vice versa.

(iii) A pair of pure 2-qutrit states $(|\psi\rangle, |\phi\rangle)$ such that neither can $|\psi\rangle$ be LOCC-transformed into $|\phi\rangle$ nor vice versa.

For each of your examples, also give short proofs that the respective LOCC transformations are (im-)possible as claimed. Ensure that your examples are simple enough so that you can prove them in 3-4 lines using Eq. (10).

2 P. (b) Is there a pair of 2-qubit states $(|\psi\rangle, |\phi\rangle)$ such that neither can $|\psi\rangle$ be LOCC-transformed into $|\phi\rangle$ nor vice versa? Either give an example or prove that such a pair does not exist.

Monogamy

5 P. Bonus Exercise 2. In this exercise, we consider a property called *monogamy of (pure state) entanglement*. First, we prove the simplest version thereof mathematically, then we interpret it physically.

2 P. (a) Suppose we have a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $|\psi\rangle = |\psi\rangle_{AB} \in \mathcal{H}$ be a pure quantum state. Assume that the reduced state $\rho_A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$ on the first subsystem is pure, $\rho_A = |\phi\rangle\langle\phi|$ for some $|\phi\rangle = |\phi\rangle_A \in \mathcal{H}_A$. Show that $|\psi\rangle_{AB}$ is a tensor product, i.e., that $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\varphi\rangle_B$ for some pure state $|\varphi\rangle = |\varphi\rangle_B \in \mathcal{H}_B$.

1 P. (b) Now let's start from the setting where we have an entangled pure state $|\psi\rangle_{AB}$. Can there exist a pure state $|\Psi\rangle_{ABC}$ where there is entanglement between AB and C such that the reduced state on AB is $|\psi\rangle_{AB}$? Explain your answer.

1 P. (c) Consider a 3-qubit GHZ state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. This is a state in which all 3 subsystems are entangled. Why does that not contradict the result of (a)?

1 P. (d) Does a similar property of monogamy hold for classical correlations? That is, if ρ_{AB} is a separable non-product state that is diagonal in the computational basis, is it true that every separable state ρ_{ABC} that is diagonal in the computational basis and satisfies $\rho_{AB} = \mathrm{Tr}_C[\rho_{ABC}]$, has to be a product state $\rho_{ABC} = \rho_{AB} \otimes \rho_C$? Either sketch a proof or give a counterexample.

Total Points: 21 (+10)

[1] This is because the transition from $|\psi\rangle$ to $|\phi\rangle$ comes about as a post-measurement state with probability p_j.