Exercise Sheet 6: LOCC and Majorization

LOCC

At the heart of entanglement theory lies the notion of LOCC (Local Operations and Classical Communication). To see why, imagine two parties that are a large distance apart from each other, say, Alice is in Berlin and Bob in New York. While they may obtain access to shared entanglement from a third party, it is unreasonable to assume that they are able to perform global operations on the state they share. However, it is perfectly conceivable that they transmit classical messages, for example, to communicate measurement results.

The goal of the next exercise is to show that if Alice and Bob are in distant labs, and share a state, any measurement on Alice's part of the state can be simulated as follows: Bob performs a measurement on his side and communicates the result to Alice, who performs a local unitary transformation. This can be proven for POVMs, but for simplicity we will restrict ourselves to projective measurements.

8 P. **Exercise 1.** We work on a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. For convenience we assume $\mathcal{H}_A \cong \mathcal{H}_B$. Consider a bipartite pure state $|\psi\rangle_{AB}$ with Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |a_i\rangle |b_i\rangle \tag{1}$$

and a projective measurement $\Pi = \{\Pi_i^A\}_i$ acting on Alice's Hilbert space.

3 P. (a) Expand Π_i^A in the basis of the Schmidt decomposition and define a projective measurement $\Gamma = \{\Gamma_i^B\}_i$ on Bob's system such that the probability p_i^B that Bob observes result i when measuring Γ is the same as the probability p_i^A that Alice observes result i when measuring Π.

Solution

We expand Alice's POVM effects as

$$\Pi_i = \sum_{kj} \Pi_i^{kj} |a_k\rangle\langle a_j|$$

and define

$$\Gamma_i = \sum_{kj} \Pi_i^{kj} |b_k\rangle\langle b_j|.$$

Note that $\Gamma = \{\Gamma_i\}_i$ forms a POVM because Γ_i differs from Π_i only by a basis change and because $\Pi = \{\Pi_i\}_i$ forms a POVM. Then,

$$\begin{aligned} p_i^B &= \langle\psi|\Gamma_i|\psi\rangle \\ &= \left(\sum_{k'} \sqrt{\lambda_{k'}} \langle a_{k'}| \otimes \langle b_{k'}|\right)\left(\sum_{kj} \Pi_i^{kj} \mathbb{I} \otimes |b_k\rangle\langle b_j|\right)\left(\sum_{j'} \sqrt{\lambda_{j'}} |a_{j'}\rangle \otimes |b_{j'}\rangle\right) \\ &= \sum_{k,k',j,j'} \sqrt{\lambda_k}\sqrt{\lambda_j} \Gamma_i^{kj} \delta_{k'j'} \delta_{kk'} \delta_{jj'} = \sum_k \lambda_k \Gamma_i^{kk} = \sum_k \lambda_k \Pi_i^{kk} = p_i^A \end{aligned}$$

An alternative way of writing this: If we define the isometry

$$I_{A\to B} \coloneqq \sum_i |b_i\rangle\langle a_i|,$$

and $I_{B\to A} = I_{A\to B}^\dagger$, we simply have that $\Gamma_i^B = I_{A\to B} \Pi_i^A I_{B\to A}$.

2 P. (b) Determine the post-measurement states $|\phi_j^A\rangle$ after Alice measures Π and obtains result j, and $|\phi_j^B\rangle$ after Bob measures Γ and obtains result j. (Note: Both of these states are defined on the whole Hilbert space AB, the superscripts serve to identify who performed the measurement).

Solution

Up to normalization $|\phi_j^A\rangle = \sum_{ik} \sqrt{\lambda_k} \Pi_j^{ik} |a_i\rangle|b_k\rangle$ and $|\phi_j^B\rangle = \sum_{ik} \sqrt{\lambda_k} \Pi_j^{ik} |a_k\rangle|b_i\rangle$.

2 P. (c) Show that $|\phi_j^A\rangle$ and $|\phi_j^B\rangle$ are equivalent up to local unitary transformations.

Hint: Recall that for a state of the form $\sum_{k\ell} C_{k\ell} |e_k\rangle|f_\ell\rangle$, with orthonormal sets $\{|e_k\rangle\}_k$ and $\{|f_\ell\rangle\}_\ell$, its Schmidt coefficients are exactly the singular values of the matrix $C = (C_{k\ell})_{k\ell}$.

Solution

Define a matrix $C^{(j)}$ with entries $C_{ik}^{(j)} = \sqrt{\lambda_k}\Pi_j^{ik}$. With this matrix, we can rewrite our states up to normalization as

$$|\phi_j^A\rangle = \sum_{ik} C_{ik}^{(j)} |a_i\rangle|b_k\rangle\,, \tag{2}$$

$$|\phi_j^B\rangle = \sum_{ik} C_{ik}^{(j)} |a_k\rangle|b_i\rangle = \sum_{ik} \left(C^{(j)\top}\right)_{ik} |a_i\rangle|b_k\rangle\,. \tag{3}$$

Thus, since $\{|a_k\rangle\}_k$ and $\{|b_\ell\rangle\}_\ell$ are orthonormal sets (remember that they come from a Schmidt decomposition), the Schmidt coefficients of $|\phi_j^A\rangle$ are (up to normalization) the singular values of $C^{(j)}$ while the Schmidt coefficients of $|\phi_j^B\rangle$ are (up to normalization) the singular values of $C^{(j)\top}$. As the matrix transpose leaves the singular values invariant, we conclude that $|\phi_j^A\rangle$ and $|\phi_j^B\rangle$ have the same Schmidt coefficients. Hence they are equivalent up to local unitary transformations (namely exactly the local unitaries that transform between the respective local Schmidt bases).

1 P. (d) Describe the LOCC protocol.

Solution

Bob performs the measurement Γ, obtains result j and communicates the result to Alice. Now both know that the global state is $|\phi_j^B\rangle$ and perform the necessary local unitaries to turn the state into $|\phi_j^A\rangle$.

Majorization

Majorization is a mathematical concept that has surprisingly far-reaching applications. Consider two vectors $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n$. We define a sorted (in descending manner) version of a vector $\boldsymbol{v}$ as $\boldsymbol{v}^\downarrow$, such that

$$v_1^\downarrow \geq v_2^\downarrow \geq \ldots \geq v_n^\downarrow. \tag{4}$$

We can write

$$\boldsymbol{v}^\downarrow = P\boldsymbol{v} \tag{5}$$

for some matrix P that permutes the entries of $\boldsymbol{v}$.

2 P. **Exercise 2.** Let us look at the vector $\boldsymbol{y} = (2, 4, 1, 3)$. What is $\boldsymbol{y}^\downarrow$ and what matrix P permutes $\boldsymbol{y}$ into $\boldsymbol{y}^\downarrow$?

Solution

We have

$$\boldsymbol{y}^\downarrow = \begin{pmatrix}4\\3\\2\\1\end{pmatrix} = \underbrace{\begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}}_{=:P} \begin{pmatrix}2\\4\\1\\3\end{pmatrix} = P\boldsymbol{y}.$$

We now say that $\boldsymbol{x}$ *majorizes* $\boldsymbol{y}$, written as $\boldsymbol{x} \succ \boldsymbol{y}$, if

$$\boldsymbol{x} \succ \boldsymbol{y} \;\Leftrightarrow\; \sum_{j=1}^{k} x_j^{\downarrow} \geq \sum_{j=1}^{k} y_j^{\downarrow} \text{ for all } 1 \leq k \leq n. \tag{6}$$

1 P. **Exercise 3.** Show that $\boldsymbol{x} = (2, 1, 0)$ majorizes $\boldsymbol{y} = (1, 1, 1)$.

Solution

Clear from evaluating the sums for all $1 \leq k \leq 3$ which are $2, 3, 3$ and $1, 2, 3$, respectively.

A central insight in the theory of majorization is that the majorization condition $\boldsymbol{x} \succ \boldsymbol{y}$ holds if and only if we can write

$$\boldsymbol{x} \succ \boldsymbol{y} \;\Leftrightarrow\; \boldsymbol{y} = \sum_j p_j P_j \boldsymbol{x} \tag{7}$$

for a probability distribution p_j over permutation matrices P_j. Birkhoff's theorem then implies that we can write

$$\boldsymbol{y} = D\boldsymbol{x} \tag{8}$$

for a *doubly-stochastic* matrix D, which is a matrix where all columns and rows are simultaneously probability distributions, i.e.

$$D \text{ is doubly-stochastic } \Leftrightarrow\; D_{ij} \geq 0 \text{ and } \sum_{i=1}^{n} D_{ij} = 1 \text{ and } \sum_{j=1}^{n} D_{ij} = 1 \text{ for all } i \text{ and } j. \tag{9}$$

The purpose of the next exercise is to uncover the role of majorization in state transformations using LOCC. Specifically, we will try to understand when we can transform a given copy of a pure bipartite quantum state $|\psi\rangle$ to another quantum state $|\phi\rangle$ using LOCC, which we write as

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle. \tag{10}$$

As a first ingredient, we extend the definition of majorization to Hermitian matrices. Let X, Y be Hermitian $n \times n$ matrices. If $\boldsymbol{\lambda}(X)$ ($\boldsymbol{\lambda}(Y)$) is the vector of eigenvalues of X (Y), then we say that X majorizes Y if their vectors of eigenvalues majorize each other, i.e.

$$X \succ Y \;\Leftrightarrow\; \boldsymbol{\lambda}(X) \succ \boldsymbol{\lambda}(Y). \tag{11}$$

4 P. **Exercise 4.** Show that $X \succ Y$ for Hermitian X and Y if and only if there exists a probability distribution p and unitary matrices U_j such that

$$Y = \sum_j p_j U_j X U_j^{\dagger}.$$

Hint: Make use of the eigendecompositions of X and Y and use Eq. (7) for the "if" part and Eq. (8) for the "only if" part.

Solution

We will first show that if X majorizes Y, we can write Y in the desired form. We will write spectral decompositions $X = V\Lambda(X)V^\dagger$ and $Y = W\Lambda(Y)W^\dagger$. As majorization implies via Eq. (7) that

$$\boldsymbol{\lambda}(Y) = \sum_j p_j P_j \boldsymbol{\lambda}(X)$$

for a probability vetor $(p_i)_i$ and for permutation matrices P_j, and since the entries of $\boldsymbol{\lambda}(X)$ and $\boldsymbol{\lambda}(Y)$ form the diagonals of $\Lambda(X)$ and $\Lambda(Y)$, respectively, we can deduce that

$$\Lambda(Y) = \sum_j p_j P_j \Lambda(X) P_j^\dagger.$$

Using $\Lambda(X) = V^\dagger X V$ and $\Lambda(Y) = W^\dagger Y W$ then yields

$$Y = \sum_j p_j W P_j V^\dagger X V P_j^\dagger W^\dagger.$$

Defining $U_j := W P_j V^\dagger$ proves the desired statement, as the matrices P_j are unitary and hence the U_j are as well.

For the reverse implication, we will again use the eigendecompositions, which allow us to transform

$$Y = \sum_j p_j U_j X U_j^\dagger$$

into

$$\Lambda(Y) = \sum_j p_j W^\dagger U_j V \Lambda(X) V^\dagger U_j W^\dagger = \sum_j p_j Q_j \Lambda(X) Q_j^\dagger,$$

where we defined the unitary matrices $Q_j := W^\dagger U_j V$. Now, we look at one diagonal entry of $\Lambda(Y)$, which can be identified with an entry of $\boldsymbol{\lambda}(Y)$. This yields

$$\begin{aligned}\lambda(Y)_i &= \sum_j p_j \sum_k [Q_j]_{ik} \lambda(X)_k [Q_j]_{ik}^* \\ &= \sum_{jk} p_j |[Q_j]_{ik}|^2 \lambda(X)_k.\end{aligned}$$

From this we can conclude that

$$\boldsymbol{\lambda}(Y) = D\boldsymbol{\lambda}(X)$$

for a matrix D with entries

$$D_{ik} = \sum_j p_j |[Q_j]_{ik}|^2$$

We are left to check that D is a doubly-stochastic matrix to prove the reverse impliciation via Eq. (8). Indeed, we have that all entries of D are non-negative and

$$\begin{aligned}\sum_i D_{ik} &= \sum_i \sum_j p_j |[Q_j]_{ik}|^2 = \sum_j p_j = 1 \\ \sum_k D_{ik} &= \sum_k \sum_j p_j |[Q_j]_{ik}|^2 = \sum_j p_j = 1,\end{aligned}$$

which concludes the proof.

6 P. **Exercise 5.** We are now ready to prove the surprising theorem that conversion of $|\psi\rangle$ into $|\phi\rangle$ under LOCC is only possible if the reduced state $\mathrm{Tr}_B[|\phi\rangle\langle\phi|]$ majorizes the reduced state $\mathrm{Tr}_B[|\psi\rangle\langle\psi|]$:

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \;\Leftrightarrow\; \mathrm{Tr}_B[|\psi\rangle\langle\psi|] \prec \mathrm{Tr}_B[|\phi\rangle\langle\phi|]. \tag{12}$$

We encourage you to have a look into the original paper that established the theorem (`https://arxiv.org/pdf/quant-ph/9811053.pdf`).

3 P. (a) Show the "only if" direction in Eq. (12) using the previous result, i.e. assume that $|\psi\rangle$ can be transformed into $|\phi\rangle$ under LOCC and show that this implies the states majorization condition. You can suppose that LOCC is realised by a measurement $M = \{M_j\}$ on Alice's side and a corresponding unitary on Bob's side. In other words, from Alice's point of view it must be the case that[1]

$$M_j \mathrm{Tr}_B[|\psi\rangle\langle\psi|] M_j^\dagger = p_j \mathrm{Tr}_B[|\phi\rangle\langle\phi|]. \tag{13}$$

Hint: Define $X_j := M_j\sqrt{\mathrm{Tr}_B[|\psi\rangle\langle\psi|]}$ such that the left hand side of the above equations is $X_j X_j^\dagger$ and use the polar decomposition of X and the fact that $\{M_j\}$ is a POVM.

Solution

Let us define $\rho_\psi := \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$ and $\rho_\phi := \mathrm{Tr}_B[|\phi\rangle\langle\phi|]$. The polar decomposition is $X_j = R_j V_j$ for a positive semi-definite matrix R_j and a unitary V_j. The matrix R_j is given by $R_j = \sqrt{X_j X_j^\dagger} = \sqrt{M_j \rho_\psi M_j^\dagger}$.
Combining the polar decomposition with the assumption yields

$$X_j = M_j\sqrt{\rho_\psi} = \sqrt{M_j \rho_\psi M_j^\dagger}V_j = \sqrt{p_j\rho_\phi}V_j,$$

where we used Eq. (13) for the last step. Now, we can write

$$\begin{aligned}
\rho_\psi &= \sqrt{\rho_\psi}\sqrt{\rho_\psi} \\
&= \sum_j \sqrt{\rho_\psi}M_j^\dagger M_j\sqrt{\rho_\psi} \\
&= \sum_j V_j^\dagger\sqrt{p_j\rho_\phi}\sqrt{p_j\rho_\phi}V_j \\
&= \sum_j p_j V_j^\dagger \rho_\phi V_j
\end{aligned}$$

which in turn implies $\rho_\psi \prec \rho_\phi$ by assertion of Exercise 4 as desired. Here, we inserted an identity $\mathbb{I} = \sum_j M_j^\dagger M_j$ for the second equality and then used the just-derived expression for X_j.

3 P. (b) Now show the "if" direction in Eq. (12) by proceeding analogously.

[1] This is because the transition from $|\psi\rangle$ to $|\phi\rangle$ comes about as a post-measurement state with probability p_j.

Solution

We run the proof of the preceding exercise backwards. By assumption and by Exercise 4, we know the existence of V_j and p_j such that

$$\rho_\psi = \sum_j p_j V_j^\dagger \rho_\phi V_j.$$

We can then define the operators

$$M_j \coloneqq \sqrt{p_j \rho_\phi} V_j \sqrt{\rho_\psi}^{-1}. \tag{14}$$

If ρ_ψ does not have full rank, it suffices to take the inverse on the support, as the POVM is only evaluated on ρ_ψ anyways and contributions outside of the support do not matter. We check that the M_j indeed fulfill the completeness relation as

$$\sum_j M_j^\dagger M_j = \sum_j \sqrt{\rho_\psi}^{-1} V_j^\dagger \sqrt{p_j \rho_\phi} \sqrt{p_j \rho_\phi} V_j \sqrt{\rho_\psi}^{-1} \tag{15}$$

$$= \sqrt{\rho_\psi}^{-1} \left(\sum_j p_j V_j^\dagger \rho_\phi V_j \right) \sqrt{\rho_\psi}^{-1} \tag{16}$$

$$= \sqrt{\rho_\psi}^{-1} \rho_\psi \sqrt{\rho_\psi}^{-1} \tag{17}$$

$$= \mathbb{I}. \tag{18}$$

Again, if the matrix ρ_ψ does not have full rank, think of the identity as to be defined on the complement of the kernel of ρ_ψ. As

$$M_j \rho_\psi M_j^\dagger = p_j \rho_\phi \tag{19}$$

by definition of M_j, we have found a LOCC scheme performing the transformation.

5 P. Bonus Exercise 1. Let's try to apply Eq. (12) a bit more concretely.

3 P. (a) Give one example for each of the following:

(i) A pair of pure 2-qubit states $(|\psi\rangle, |\phi\rangle)$ such that $|\psi\rangle$ can be LOCC-transformed into $|\phi\rangle$ but not vice versa.

(ii) A pair of pure 2-qubit states $(|\psi\rangle, |\phi\rangle)$ such that $|\psi\rangle$ can be LOCC-transformed into $|\phi\rangle$ and vice versa.

(iii) A pair of pure 2-qutrit states $(|\psi\rangle, |\phi\rangle)$ such that neither can $|\psi\rangle$ be LOCC-transformed into $|\phi\rangle$ nor vice versa.

For each of your examples, also give short proofs that the respective LOCC transformations are (im-)possible as claimed. Ensure that your examples are simple enough so that you can prove them in 3-4 lines using Eq. (12).

Solution

We first give an example for (i). Take $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\phi\rangle = |00\rangle$. The states are already in Schmidt decomposition form, the Schmidt values are $\lambda_1^\psi = \frac{1}{2} = \lambda_2^\psi$ and $\lambda_1^\phi = 1$, $\lambda_2^\phi = 0$. Clearly, $\lambda_1^\psi < \lambda_1^\phi$ and $\lambda_1^\psi + \lambda_2^\psi = 1 = \lambda_1^\phi + \lambda_2^\phi$. So, by Eq. (12), $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ but not vice versa.

For (ii), the easiest example to pick is $|\psi\rangle = |\phi\rangle$ for any 2-qubit state $|\psi\rangle$. In fact, it is not hard to see that any pair satisfying the desired property has to be a pair of two 2-qubit states with the same Schmidt values (up to reordering).

Now we give an example for (iii), taken from Nielsen's paper (`https://arxiv.org/pdf/quant-ph/9811053.pdf`). Here, we take

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \sqrt{\frac{2}{5}}|11\rangle + \frac{1}{\sqrt{10}}|22\rangle \tag{20}$$

$$|\phi\rangle = \sqrt{\frac{3}{5}}|00\rangle + \frac{1}{\sqrt{5}}|11\rangle + \frac{1}{\sqrt{5}}|22\rangle\,. \tag{21}$$

The states are already in Schmidt decomposition form, we can read off the Schmidt values and notice:

- $\lambda_1^\psi = 1/2 < 3/5 = \lambda_1^\phi$, so $\mathrm{Tr}_B[|\phi\rangle\langle\phi|] \not\prec \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$. Hence, $|\phi\rangle$ cannot be LOCC-transformed into $|\psi\rangle$.
- $\lambda_1^\psi + \lambda_2^\psi = 1/2 + 2/5 = 9/10 > 4/5 = 3/5 + 1/5 = \lambda_1^\phi + \lambda_2^\phi$, so $\mathrm{Tr}_B[|\psi\rangle\langle\psi|] \not\prec \mathrm{Tr}_B[|\phi\rangle\langle\phi|]$. Hence, $|\psi\rangle$ cannot be LOCC-transformed into $|\phi\rangle$.

2 P. (b) Is there a pair of 2-qubit states $(|\psi\rangle, |\phi\rangle)$ such that neither can $|\psi\rangle$ be LOCC-transformed into $|\phi\rangle$ nor vice versa? Either give an example or prove that such a pair does not exist.

Solution

No, such a pair does not exist. This can be seen as follows: Let $|\psi\rangle, |\phi\rangle$ be 2-qubit states with Schmidt values $\lambda_1^\psi, \lambda_2^\psi$ and $\lambda_1^\phi, \lambda_2^\phi$. W.l.o.g. these Schmidt values are already in non-increasing order. Assume that $|\psi\rangle$ cannot be LOCC-transformed into $|\phi\rangle$. As $\lambda_1^\psi + \lambda_2^\psi = 1 = \lambda_1^\phi + \lambda_2^\phi$, by Eq. (12) this implies $\lambda_1^\psi > \lambda_1^\phi$. However, together with $\lambda_1^\psi + \lambda_2^\psi = 1 = \lambda_1^\phi + \lambda_2^\phi$, this then implies that $|\phi\rangle$ can be LOCC-transformed into $|\psi\rangle$ according to Eq. (12).

Monogamy

5 P. **Bonus Exercise 2.** In this exercise, we consider a property called *monogamy of (pure state) entanglement*. First, we prove the simplest version thereof mathematically, then we interpret it physically.

2 P. (a) Suppose we have a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $|\psi\rangle = |\psi\rangle_{AB} \in \mathcal{H}$ be a pure quantum state. Assume that the reduced state $\rho_A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$ on the first subsystem is pure, $\rho_A = |\phi\rangle\langle\phi|$ for some $|\phi\rangle = |\phi\rangle_A \in \mathcal{H}_A$. Show that $|\psi\rangle_{AB}$ is a tensor product, i.e., that $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\varphi\rangle_B$ for some pure state $|\varphi\rangle = |\varphi\rangle_B \in \mathcal{H}_B$.

Solution

Consider the Schmidt decomposition of $|\psi\rangle_{AB}$:

$$|\psi\rangle_{AB} = \sum_{j=1}^{\min\{d_A,d_B\}} \sqrt{\lambda_j}|e_j\rangle|f_j\rangle\,, \tag{22}$$

where $\lambda_j \geq 0$ for all j, $\sum_j \lambda_j = 1$, $\{|e_j\rangle\}_j$ is an ONB for $\mathcal{H}_A$ and $\{|f\rangle_j\}_j$ is an ONB for $\mathcal{H}_B$. From the Schmidt decomposition, we can read off the reduced density matrix:

$$\mathrm{Tr}_B[|\psi\rangle\langle\psi|] = \sum_{j=1}^{\min\{d_A,d_B\}} \lambda_j|e_j\rangle\langle e_j|\,. \tag{23}$$

By assumption, this reduced density matrix equals the pure state $|\phi\rangle\langle\phi|$. As $|\phi\rangle\langle\phi|$ has rank 1, this is only possible if there exists $1 \leq i \leq \min\{d_A, d_B\}$ such that $\lambda_i = 1$, $|e_j\rangle = |\phi\rangle$, and $\lambda_j = 0$ for $j \neq i$. Plugging this back into the Schmidt decomposition of $|\psi\rangle$, we get

$$|\psi\rangle_{AB} = |\phi\rangle|f_i\rangle\,, \tag{24}$$

so $|\psi\rangle$ indeed is a tensor product state (with the $|\varphi\rangle$ from the question being $|f_i\rangle$).

1 P. (b) Now let's start from the setting where we have an entangled pure state $|\psi\rangle_{AB}$. Can there exist a pure state $|\Psi\rangle_{ABC}$ where there is entanglement between AB and C such that the reduced state on AB is $|\psi\rangle_{AB}$? Explain your answer.

Solution

In words, the result proved in (a) says the following: If the reduced density matrix on some subsystem is pure, then that subsystem does not have any quantum correlations with any other system. As a consequence of (a), if we have a bipartite entangled pure state $|\psi\rangle_{AB}$, then it cannot be entangled with any other quantum system C via an overall pure quantum state $|\Psi\rangle_{ABC}$, since that would violate the tensor product form. In other words: Correlations in a pure quantum state are "monogamous", they cannot be shared with further systems.

1 P. (c) Consider a 3-qubit GHZ state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. This is a state in which all 3 subsystems are entangled. Why does that not contradict the result of (a)?

Solution

Any 2-qubit reduced density matrix of the GHZ state equals $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. This is a separable mixed state. In particular, as this reduced density matrix is a mixed state, we cannot apply (a). Moreover, note that this reduced density matrix is not entangled. Thus, while all three systems are in an entangled state, if we "throw away" any of the three subsystems, we completely destroy the entanglement.

1 P. (d) Does a similar property of monogamy hold for classical correlations? That is, if ρ_{AB} is a separable non-product state that is diagonal in the computational basis, is it true that every separable state ρ_{ABC} that is diagonal in the computational basis and satisfies $\rho_{AB} = \mathrm{Tr}_C[\rho_{ABC}]$, has to be a product state $\rho_{ABC} = \rho_{AB} \otimes \rho_C$? Either sketch a proof or give a counterexample.

Solution

No, classical correlations are very much not monogamous. There are different ways of seeing this (e.g., if you clone a classical system multiple times, you get many perfectly correlated copies). Here's a simple counterexample phrased in the notation that we've been using all along: Consider a (diagonal, hence classical) mixed state

$$\rho_{ABC} = \sum_{i=1}^{d} p_i |i\rangle\langle i| \otimes |i\rangle\langle i| \otimes |i\rangle\langle i| \,, \tag{25}$$

where $(p_i)_i$ is a probability vector and we for simplicity assumed that $\mathcal{H}_A, \mathcal{H}_B$, and $\mathcal{H}_C$ are all isomorphic to $\mathbb{C}^d$. This state describes three perfectly correlated classical random variables. The reduced density matrix on the first two subsystems is

$$\rho_{AB} = \sum_{i=1}^{d} p_i |i\rangle\langle i| \otimes |i\rangle\langle i| \,. \tag{26}$$

This state describes two perfectly correlated random variables on the A and B systems. Here, we have perfect correlation between A and B, but in general these two subsystems are further correlated with C (since ρ_{ABC} is not a tensor product if $(p_i)_i$ has more than one non-zero entry). This shows by example that classical correlations can be shared between more than just two systems.

Total Points: 21 (+10)