Exercise Sheet 7: Channel capacities

Capacities of classical channels

Shannon's noisy channel coding theorem states that the capacity of a (noisy) classical channel T is given by the maximum input-output mutual information:

$$C(T) = \max_{X, p_X} I(X:Y)\,,$$

with the input-output mutual information

$$I(X:Y) = H(X) + H(Y) - H(X,Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}.$$

Here X is an input random variable with distribution p_X, and where Y is the random variable describing the output of the channel T with input X. We will determine the capacity of two example channels.

The first channel we treat is the *binary symmetric channel* or *bitflip channel* T_{BSC}. With probability p, the input bit is flipped ($0 \to 1$ and $1 \to 0$), and with probability $1-p$ it is transmitted without error. We can also formulate this in terms of the following conditional probabilities of the outputs given the inputs:

$$T_{\mathrm{BSC}}\colon \quad \begin{aligned} \mathbb{P}[0\,|\,0] &= \mathbb{P}[1\,|\,1] = 1-p, \\ \mathbb{P}[1\,|\,0] &= \mathbb{P}[0\,|\,1] = p. \end{aligned}$$

4 P. Exercise 1.

3 P. (a) Determine the capacity of the binary symmetric channel $C(T_{\mathrm{BSC}})$ in terms of the binary entropy (which is the entropy of a coin/Bernoulli random variable with probability p)

$$H_2(p) := -p \log p - (1-p)\log(1-p). \tag{1}$$

For the rest of the sheet, we assume logarithms to be base 2.

Hint: It may be useful to expand $H(Y|X)$ as $\sum_x p(x) H(Y|X=x)$.

1 P. (b) The binary symmetric channel T_{BSC} is a classical communication channel. Write down a quantum channel that generalizes it (in the sense that it acts similarly on computational basis states.)

At the output of the binary symmetric channel, it is not possible to detect if an error occurred or not. If we instead got an output "error", every time an error occurred, our lives might get easier. The channel describing this is the so-called *binary erasure channel* T_{E}. Mathematically, we describe it by adding the "error state" e to the possible states of the system (besides 0 and 1). Formally, the binary erasure channel outputs "error" (i.e. the state e) with probability p and with probability $1-p$ transmits the input without error, leading to the following conditional probabilities:

$$T_{\mathrm{E}}\colon \quad \begin{aligned} \mathbb{P}[0\,|\,0] &= \mathbb{P}[1\,|\,1] = 1-p, \\ \mathbb{P}[e\,|\,0] &= \mathbb{P}[e\,|\,1] = p. \end{aligned}$$

9 P. Exercise 2. Let us now determine the capacity of T_{E}.

1 P. (a) We first introduce another random variable Z. Namely, we let $Z = \mathbb{1}_E$ be the indicator random variable for the event $E = \{Y = e\}$ that an erasure error occurs. What is $\mathbb{P}[Z=1]$ in terms of p?

2 P. (b) Given this new random variable Z (note that its complement is $\neg E = \{Y \neq e\}$), we can write

$$H(Y) = H(Y, Z) = H(Z) + H(Y \mid Z), \tag{2}$$

because Z is a deterministic function of Y. Use the above expansion to show that $H(Y) = H_2(p) + (1-p)H_2(\mathbb{P}[X = 1])$, where H_2 is the binary entropy defined in Eq. (1).

Hint: Use that $\mathbb{P}[Y = y \mid Y \neq e] = \mathbb{P}[X = y]$.

2 P. (c) Compute the conditional entropy $H(Y \mid X)$.

2 P. (d) Now proceed similarly to Exercise 1 to determine the channel capacity of the erasure channel.

1 P. (e) Compare the channel capacities of T_{BSC} and T_{E}. Give an approximate numerical value p_0 such that for $p < p_0$ you would rather communicate via T_{E} than via T_{BSC}.

1 P. (f) The erasure channel T_{E} is a classical communication channel. Write down a quantum channel that generalizes it (in the sense that it acts similarly on computational basis states.)

Classical capacity of a quantum channel

In the lecture, we saw two alternative characterizations of the classical channel capacity of a quantum channel $\mathcal{E}$, which is given by the Holevo-information $\chi(\mathcal{E})$. We want to establish the equivalence of these expressions.

To this end, recall the definition of the quantum mutual information of a bi-partite quantum system in a state ρ_{AB}

$$I(A : B)_{\rho_{AB}} := S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{3}$$

The Holevo information of channel can be defined using the following scheme: Alice encodes the information of a classical random variable X taking values in $\mathcal{X}$ with probability distribution p_X into a quantum state using a set of states $\{\rho_x\}_{x\in\mathcal{X}}$. To keep track of the classical random variable but formulating everything quantum mechanically, we think of Alice keeping a "notebook" of the information she encoded which we can model as storing that information in another register N using an orthogonal basis $\{|x\rangle\}_{x\in\mathcal{X}}$. From this "notebook" register N, the classical information of X can be completely recovered. Altogether, Alice prepares the bi-partite state

$$\rho_{NA} = \sum_x p_X(x) |x\rangle\langle x|_N \otimes \rho_A^x. \tag{4}$$

Then, the state in system A is sent to Bob using the channel $\mathcal{E}$. Thus, we end up with a final state shared between Alice's notebook and Bob

$$\rho_{NB} = \sum_x p_X(x) |x\rangle\langle x|_N \otimes \mathcal{E}[\rho_A^x]_B. \tag{5}$$

We can now ask for the mutual information between the variable X encoded in N and Bob's output of the channel. Analogously to the classical result, maximizing the mutual information over all possible input variables X and encodings yields the capacity of the quantum channel to transmit classical information, i.e.

$$\chi(\mathcal{E}) = \max_{(X, p_X, \{\rho^x\})} I(N : B)_{\rho_{NB}}. \tag{6}$$

4 P. **Exercise 3.** Show that the Holevo information can be equivalently computed as the largest difference between the entropy of the expected output state and the expected entropy of the output state:

$$\chi(\mathcal{E}) = \max_{(X,p_X,\{\rho^x\})} \left\{ S\left(\mathcal{E}\left[\sum_x p_X(x)\rho^x\right]\right) - \sum_x p_X(x) S\left(\mathcal{E}\left[\rho^x\right]\right) \right\}. \tag{7}$$

Hint: The result of Bonus Exercise 1 of sheet 5 will be useful.

Let's take another perspective on the Holevo information. The quantum relative entropy between two states ρ and σ is defined as

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log\rho - \log\sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{else} \end{cases}.$$

Here, the support of a Hermitian operator is the orthogonal complement of its kernel or, equivalently, the span of the eigenstates associated its non-zero eigenvalues. The quantum relative entropy plays a crucial role in quantum information theory and is related to many quantum entropic quantities. Here, we prove one such relationship, namely between the quantum relative entropy and the Holevo information.

4 P. **Bonus Exercise 1.** Prove the following equality:

$$\chi(\mathcal{E}) = \max_{(X,p_X,\{\rho^x\})} \mathbb{E}_X\left[D\left(\mathcal{E}[\rho^X] \,\|\, \mathbb{E}_X[\mathcal{E}[\rho^X]]\right)\right].$$

Hint: You first have to show that the relative entropies involved are not infinite.

Recap

In this exercise, we will also learn to use the three important representations of channels that we have encountered so far – Choi-Jamiołkowski, Kraus, and Stinespring – in concrete examples rather than just abstractly. Recall the qubit noise channels

$$F_\epsilon(\rho) := \epsilon X\rho X + (1-\epsilon)\rho$$
$$D_\epsilon(\rho) := \epsilon\,\mathrm{Tr}[\rho]\frac{\mathbb{I}}{2} + (1-\epsilon)\rho$$

where $\epsilon \in [0,1]$, from the last sheet.

6 P. **Exercise 4.** We derive different representations for our two example channels (recall Exercise 4 in Exercise sheet 5).

3 P. (a) Give the Choi-Jamiołkowski state, a Kraus representation, and a Stinespring representation for F_1. Explicitly write down the environment system Hilbert space and the Stinespring dilation unitary.

Hint: For the Stinespring dilation, don't just shut up and calculate. Stop and consider: What is the smallest possible additional Hilbert space which you need to make the evolution unitary?

3 P. (b) Give the Choi-Jamiołkowski state, a Kraus representation, and a Stinespring representation for D_1. Explicitly write down the environment Hilbert space and the Stinespring dilation isometry (no need to complete it to a unitary).

3 P. **Exercise 5.** Explain how to extend your results achieved in Exercise 4 from $\epsilon = 1$ to arbitrary $\epsilon \in [0, 1]$. *Hint: Don't start from scratch. Rather, determine the respective representations for the part of the channel that you are still missing, then appropriately combine that with your results from Exercise 4.*

Total Points: 26 (+4)