Contents

Chapter 8

Particles in electro-magnetic fields

This will be a very brief chapter, concerned with what happens if we put particles with mass m into an electro-magnetic field.

8.1 Hamilton operator of spinless particles

The Hamilton operator of a particle with mass m and charge e in a electro-magnetic field with vector potential A and scalar potential Φ is given by the following operator

Hamilton operator of a charged particle in the electro-magnetic field:

$$H = \frac{1}{2m}\left(P^2 - \frac{e}{c}A\right)^2 + e\Phi. \tag{8.1}$$

In the position representation, the Schroedinger equation hence reads

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A(x,t)\right)^2 + e\phi(x,t)\right)\psi(x,t). \tag{8.2}$$

For the mixed term in the quadratic expression on the right hand side one finds

$$-\frac{\hbar e}{2imc}(\nabla\cdot A + A\cdot\nabla)\psi = \frac{ie\hbar}{mc}\nabla\psi, \tag{8.3}$$

using the Coulomb gauge $\nabla A = 0$. In this gauge, we hence find

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{i\hbar e}{mc}A\cdot\nabla\psi + \frac{e^2}{2mc^2}A^2\psi + e\Phi\psi. \tag{8.4}$$

8.2 Constant magnetic field

8.2.1 Hamiltonian of a particle in a constant magnetic field

For a constant magnetic field B, one can write

$$A = -\frac{1}{2}(x \times B), \tag{8.5}$$

as one can easily verify by considering $\nabla \times A$. The second term in Eq. (8.4) takes the form

$$\begin{aligned} \frac{i\hbar e}{mc} A \cdot \nabla\psi &= -\frac{1}{2}\frac{i\hbar e}{mc}(x \times B) \cdot \nabla\psi \\ &= \frac{i\hbar e}{2mc}(x \times \nabla) \cdot B\psi \\ &= -\frac{e}{2mc} L \cdot B\psi, \end{aligned} \tag{8.6}$$

where L is the angular momentum operator. The third term in Eq. (8.4) becomes

$$\begin{aligned} \frac{e^2}{2mc^2} A^2\psi &= \frac{e^2}{8mc^2}(x \times B)^2\psi \\ &= \frac{e^2}{8mc^2}(x^2 B^2 - (x \cdot B)^2)\psi. \end{aligned} \tag{8.7}$$

The second terms is associated with paramagnetism, the third one with diamagnetism. The third term is usually negligible for atoms when $\langle L_3 \rangle \neq 0$, so whenever the second term contributes.

8.2.2 Normal Zeeman effect

Denote with $H^{(0)}$ the Hamiltonian of the hydrogen atom that we consider earlier in this lecture. We now apply a magnetic field to the system. Neglecting the third term, the Hamiltonian hence becomes

$$H = H^{(0)} + \frac{e}{2mc} B L_3, \tag{8.8}$$

where again the B field has been aligned with e_3, so with the z-axis. The eigenvectors of $H^{(0)}$ we know: We denoted them with $|\psi_{n,l,m}\rangle$. But it is easy to see that these eigenvectors of $H^{(0)}$ are still eigenvectors of H, with eigenvalues

$$E_{n,l,m} = -\frac{m e_0^4}{2\hbar^2}\frac{1}{n^2} + \hbar\omega_L m, \tag{8.9}$$

where

$$\omega_L = -\frac{eB}{2mc} = \frac{e_0 B}{2mc} \tag{8.10}$$

is the *Larmor frequency*. Hence, one would expect the magnetic field to lift the $(2l+1)$-fold degeneracy in $2l+1$ equidistant energy levels. This is not what one observes for the hydrogen atom, however: Instead, one observes a splitting as if the angular momentum took half integer values. (For other atoms, one can see the version of the Zeeman effect for historical reasons called "normal Zeeman effect").

8.2.3 Anomalous Zeeman effect

At this point, this no longer comes as a surprise, as we know that the electron has both orbital angular momentum as well as a spin. The total Hamiltonian of the hydrogen atom respecting spin is given by the following Hamiltonian:

Hamiltonian of the hydrogen atom in a constant magnetic field:

$$H = H^{(0)} - \frac{e}{2mc}(L + gS) \cdot B. \tag{8.11}$$

Here, $g = 2$ to a very good approximation, in fact,

$$g = 2.002319304718(564). \tag{8.12}$$

This constant is called the *Lande factor* or the *gyromagnetic ratio*. This Hamiltonian explains then the splitting of the levels of the hydrogen atom in a constant magnetic field. The anomalous Zeeman effect is hence due to the contribution of the spin degree of freedom. It turns out that relativistic corrections also contribute to some extent, but we will not discuss this here.

8.3 Concluding remarks

There is a lot more to say about particles in electro-magnetic fields, but we will leave it at that at this point. We end this brief chapter by a few concluding remarks, though.

- Maybe the most remarkable of all elementary effects is the *Aharonov-Bohm effect* which describes the motion of an electron in a situation where $B(x) = 0$ in a large region, as it is realized outside of an (in an idealization) infinite coil. It turns out that if one performs an interference experiment in a way that the particle is at all times in the region for which $B(x) = 0$, the particle still "feels" the nonvanishing vector potential A.

- A similar effect is that in *superconductors*, the *flux* can only take discrete values.

- The *free motion* of electrons is actually also quite intricate and one encounters *Landau levels*.