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Chapter 9

Bell's inequalities and absolute randomness

9.1 Introduction

On several occasions, we have said that the randomness of quantum mechanics was absolute. There is no internal mechanism that can be held responsible for the randomness of the measurement process. While this is how quantum mechanics is usually taught, one might argue: Well, even if we do not know this mechanism, it could still exist. At the end of the day, quantum mechanics could be just a statistical theory, possible reflecting a fine-grained classical mechanics on a smaller level. How would we know? And to argue whether quantities that can be attributed to quantum systems that determine measurement outcomes, so called *local hidden variables*, exist or not, seems largely a matter of taste. Or at least a metaphysical question.

In this chapter, we show that, surprisingly, this is not a true statement: Whether local hidden variables exist or not is actually *not* a metaphysical question. In fact, it can be decided in experiments.

This surprising insight is due to John Bell, who in his first profession was leading the theory efforts at CERN and only in his second job was thinking about the foundations of quantum mechanics.

9.2 The historical setting

Quantum mechanics emerged with remarkable speed around 1926, when decades of speculation gave way to the successful construction of a coherent theory, ushering in a golden age of groundbreaking discoveries. In much the same

way, the origin of a specific branch of quantum theory—the theory of entanglement—can be traced to a precise moment. That moment is 1935, marked by the publication of the Einstein–Podolsky–Rosen (EPR) paper and, inspired by it, Schrödinger's article in which he introduced the concept of the "entangled state" (verschränkter Zustand).

Although both Einstein and Schrödinger had played central roles in shaping quantum theory, they each voiced profound dissatisfaction with what Schrödinger called the "present situation of quantum theory." Their concerns were largely dismissed by some younger physicists as the complaints of aging scientists unable to adapt to new ways of thinking. Bohr's response to the EPR paper—essentially a rejection of the problem rather than a direct resolution—was nevertheless celebrated as definitive, and the physics community largely moved on. With the benefit of hindsight, however, it is clear that Einstein was grappling with the most radical break from classical physics introduced by quantum mechanics: entanglement. While not all of his conclusions were correct, his insight was fundamentally sound. He correctly identified entanglement as the point at which quantum theory departs most profoundly from classical intuition.

At the time of its publication, the significance of the EPR paper was barely recognized by the physics community. It attracted little attention and was largely set aside, only to be rediscovered decades later as one of the most profound contributions to quantum theory. This delayed impact can be traced, at least in part, to features intrinsic to the paper itself. First, the EPR argument revolved around the notion of the completeness of quantum mechanics—a concern that felt abstract and premature to a community exhilarated by the theory's extraordinary empirical successes. With quantum mechanics delivering one triumph after another, questions about its ultimate conceptual foundations seemed like problems for a distant future. Second, the technical formulation of the argument appeared mathematically awkward, even artificial. The state considered by Einstein, Podolsky, and Rosen involved continuous variables and idealized harmonic oscillators, resulting in a highly singular construction. To this day, physicists still debate how to give the original formulation a fully rigorous mathematical footing. As a consequence, the perfect correlations between distant measurements—central to the EPR argument—were not even strictly exact within the original setup. This technical weakness was later remedied by David Bohm, who reformulated the argument using spin systems—what we now call qubits—rendering the correlations both precise and transparent. Yet the truly decisive transformation came in 1964, when John Bell made a conceptual leap of extraordinary significance. Bell showed that the EPR dilemma could be cast as a set of explicit assumptions, leading directly to experimentally testable—and falsifiable—predictions. The importance of this insight can scarcely be overstated. For the first time, it became possible to experimentally rule out not merely a specific physical theory, but an entire classical worldview that had shaped the formulation of scientific theories for centuries. Entanglement was no longer a philosophical curiosity—it had become an empirical fault line separating classical intuition from the quan-

tum reality.

9.3 Preliminaries

And here come some basic preliminaries to the argument. It is important to note that it is not about quantum mechanics, but in fact about any physical theory.

9.3.1 Correlation tables

Bell-type inequalities are always concerned with correlations between measurements performed at two or more spatially separated sites, often referred to as parties. For clarity, it is convenient to imagine that the experiments at these sites are carried out by physicists—traditionally called Alice and Bob in the bipartite (two-party) scenario. Each party receives a particle (or, more generally, a subsystem) emitted from a common source and performs a local measurement on it. The fundamental objects of interest in this framework are the joint probabilities associated with the measurement outcomes obtained in this way. A typical measured probability will be denoted by

$$P(a_1, b_2|A_1, B_2) \tag{9.1}$$

Here, the symbols following the vertical bar specify the measurement devices employed—namely, device A_1 used by Alice and device B_2 used by Bob—while the symbols preceding the bar indicate the corresponding measurement outcomes: a_1 denotes a possible outcome of A_1, and b_2 a possible outcome of B_2. For simplicity, we will assume throughout that each measurement has only a finite number of possible outcomes. The complete set of these probabilities constitutes the fundamental experimental data, which we may collectively refer to as the correlation table. Let us keep it simple:

Two apparata, two outcomes: We assume that both Alice and Bob have two apparata each, A_1 and A_2 as well as B_1 and B_2. And each such apparatus has two outcomes, ± 1.

It is no accident that this is like at the boxes at the beginning of the lecture.

9.3.2 Properties of correlation tables

Since these correlations reflect *probabilities* in one way or the other, they satisfy

$$P(a, b|A, B) \geq 0 \tag{9.2}$$

for all A, B and a, b, and the probabilities for each setup (A, B) they have to add up to 1 as

$$\sum_{a,b=\pm 1} P(a, b|A, B) = 1. \tag{9.3}$$

This is nothing but saying that we have some probability distribution. A second kind of constraint is more subtle: Let us denote the marginals with

$$P(a|A,B) := \sum_{b=\pm 1} P(a,b|A,B). \tag{9.4}$$

These are the probabilities measured by Alice in a given setup (A,B), disregarding the outcome of B.

No-signalling condition: We assume that

$$\sum_b P(a,b|A,B) = P(a|A) \tag{9.5}$$

is independent of B, and similarly for A.

This is extremely natural: Do not forget that we have made no assumptions on how close A and B are to each other. In particular, they can we so far away to each other that by virtue of relativity, there is no way that choices of apparata made by Bob can causally influence the outcome of Alice. There cannot be "signals" faster than light transmitted. While this is an assumption, it is an extremely soft one, as a violation thereof immediately leads to the possibility of interacting with the own past.

9.3.3 Local hidden variables

Before we formalize the argument, it is important to stress that the way local hidden variables are defined is an extremely forgiving and general definition. It just refers to the situation that there are some internal degrees of freedom, no matter how simple or complicated, that dictate locally measurement outcomes in quantum mechanics. For lack of a better term, let us call these variables λ which takes values in a space Λ. They could be from a very high dimensional set, they could be very complicated. Think of them as some internal structure of, say, atoms, that determines measurement outcomes. Note that we can even assume these degrees of freedom to be volatile and not measurable in practice. We just make the very mild assumption that they exist.

We assume that the systems sent to Alice and Bob are fully characterized by a variable λ, containing all the information required to predict—or at least assign probabilities to—their responses to any possible measurement. Consequently, for any measurement device A used by Alice and any outcome a associated with that device, one can define a corresponding response probability function $\lambda \mapsto \chi_A(a,\lambda)$. A local hidden variable model is now one where the above probabilities can be written in this form.

Local hidden variable model:

$$P(a,b|A,B) = \int d\lambda f(\lambda)\chi_A(a,\lambda)\chi_B(b,\lambda), \tag{9.6}$$

for some probability distribution f on Λ.

If this may look a bit clumsy, it is not because we are making strong assumptions: The opposite is the case. All we are doing is to say that in one way or the other, for the measurement outcomes some quantities λ can be made responsible, and this dependence can be probabilistic or complicated. This is clearly a good assumption. So far, so good.

9.4 Deriving the Bell inequality

What we will now do it to derive what is called a Clauser-Horne-Shimony-Holt inequality. This is a particularly transparent form of a Bell inequality. Let us remind ourselves that the setting is made simple as we always have two outcomes ± 1. From the above response probabilities $\chi_A(a,\lambda)$, with $a = \pm 1$, we form the mean value of the random variable a given the hidden variable λ

$$\hat{a}(\lambda) := \chi_A(+1,\lambda) - \chi_A(-1,\lambda). \tag{9.7}$$

From these, and suppressing the argument λ, we can construct the correlation function

$$\mathcal{B}(\lambda) := \frac{1}{2}\left(\hat{a}_1(\hat{b}_1 + \hat{b}_2) + \hat{a}_2(\hat{b}_1 - \hat{b}_2)\right). \tag{9.8}$$

Now comes an interesting observation. We claim that $\mathcal{B}$ satisfies the pointwise inequality given by

$$|\mathcal{B}(\lambda)| \leq 1. \tag{9.9}$$

This is actually easy to see. The extreme values are attained, when each $\hat{a}(\lambda), \hat{b}(\lambda)$ is extremal, i.e., takes the value ± 1. But then $2\mathcal{B}(\lambda)$ is an even integer, and since $2\mathcal{B}(\lambda) = 4$ requires

$$\hat{a}_1(\lambda) = \hat{b}_1(\lambda) = \hat{b}_2(\lambda) = \hat{a}_2(\lambda) = -\hat{b}_2(\lambda), \tag{9.10}$$

leading to a contradiction. Hence, we must have $2\mathcal{B}(\lambda) \leq 2$. Since $\mathcal{B}$ is pointwise bounded, the same holds true for its expectation,

$$\beta := \int d\lambda f(\lambda)\mathcal{B}(\lambda), \tag{9.11}$$

so that

$$|\beta| \leq 1. \tag{9.12}$$

Where is this all leading, one might think? We are almost there. Let us introduce the expectation values

$$E(A,B) = \sum_{a,b=\pm 1} abP(a,b|A,B). \tag{9.13}$$

Written in terms of these expectation values, the inequality $|\beta| \leq 1$ becomes the following inequality.

Clauser-Horne-Shimony-Holt (CHSH) inequality:

$$|E(A_1,B_1) + E(A_1,B_2) + E(A_2,B_1) - E(A_2,B_2)| \leq 1. \tag{9.14}$$

The significance of this can hardly be overestimated. These expectation values can be measured in actual correlation experiments! In fact, they very much resemble the settings we have been discussing in the first lecture concerning the boxes. One makes many runs of identically prepared systems, takes data and estimates the above expectation values. And no matter how the precise functioning is, and no matter what one specifically does: Either way, the outcomes must satisfy the CHSH inequality. The assumptions have been extremely weak, basically only arguing that some local hidden variable λ can in one way or the other be held responsible for the outcomes.

9.5 Violation of the Bell inequalities

Quantum mechanics violates the Bell inequalities: The trouble is, quantum mechanics violates the CHSH inequalities.