Statistical Mechanics WS 2013/14 Sheet 1

Please hand in your solutions (in pairs) before the Monday lecture at 10:15.

Problem 1 : Binomial and Poisson Distributions (20 points)

The binomial distribution $P_N(m; p)$, as covered in the lecture, gives ***exact*** solutions of mean $\langle m \rangle$ and variance Δm^2 as functions of N (total trial number per one system in the ensemble) and p (probability for an event occurring at one trial), respectively.

a) (4 points) Calculate $\langle m^3 \rangle$.

b) (i) (2 points) Express Δm^2 as a function of $\langle m \rangle$ and N only.
(ii) (3 points) Express $\langle m \rangle$ using Δm^2 and N only. What is the condition for $\langle m \rangle$ to be real ($\in \mathbb{R}^1$)?

c) (i) (2 points) Find values of $P_{10}(m; 0.4)$ for $m = 0, 1, \cdots, 5$.
(ii) (2 points) Find values of $P_{10}(m; 0.01)$ for $m = 0, 1, \cdots, 5$.
(iii) (2 points) Find values of the Poisson distribution using the same values of N and p used in (ii), for $m = 0, 1, \cdots, 5$.

d) (5 points) Calculate the second moment and variance of m for the Poisson distribution $W(\lambda; m)$ with $\lambda = Np$.

Problem 2 : Multinomial Distribution (20 points)

The multinomial distribution,

$$\mathcal{P}_N(\{m_j; p_j\}) = \frac{N!}{m_1! \cdots m_k!} p_1^{m_1} \cdots p_k^{m_k},$$

is a generalization of the binomial distribution $P_N(m; p)$, for more than two random variables $\{m_1, m_2, \cdots, m_k\}$ with their probabilities $\{p_1, p_2, \cdots, p_k\}$, where $\sum_{j=1}^{k} m_j = N$ and $\sum_{j=1}^{k} p_j = 1$.

a) (5 points) Show that the variance is $\Delta m_j^2 = Np_j(1 - p_j)$.

b) (5 points) Show that the covariance is $\langle \Delta m_j \Delta m_n \rangle = \langle m_j m_n \rangle - \langle m_j \rangle \langle m_n \rangle = -Np_j p_n$.

c) (5 points) Consider 3-dimensional random walk with probabilities for going up, down, front, back, left and right are all equal. The random walker always moves a distance Δl in one of the six directions at each step. Calculate the mean squared displacement $\langle (\mathbf{r} - \langle \mathbf{r} \rangle)^2 \rangle$ after N steps. You may use the variance and the covariance in a) and b).

d) (5 points) The random walker finally gets tired. The random walker does not move and takes a rest with the same probability as it moves in each of the six directions at each step. Calculate the mean squared displacement $\langle (\mathbf{r} - \langle \mathbf{r} \rangle)^2 \rangle$ after N steps. You may use the variance and the covariance in a) and b).

Problem 3 : Characteristic Function (20 points)

a) For a probability density function $w(x) = Ae^{-|x-b|}$ where A and b are positive real numbers, respectively,
(i) (2 points) Find the value of A.
(ii) (4 points) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and $\langle (x - \langle x \rangle)^2 \rangle$.

b) If we measured, from experiments, $\langle x^m \rangle = a^m$ where a is a real number,
(i) (6 points) What would be the probability density functoin $w(x)$ and the characteristic function $g(k)$, respectively?
(ii) (6 points) Find the value of variance $\langle (x - \langle x \rangle)^2 \rangle$.
(iii) (2 points) Briefly explain (in one sentence) the physical meaning of this variance in terms of fluctuations and probability density.

(see next page for Problem 4.)

Problem 4 : Central Limit Theorem (20 points)

a) (i) (15 points) Show that if all random variables $y_1, y_2, \cdots, y_N$ are independent and have the Lorentzian distribution $L(y_j) = \frac{1}{\pi}\left[\frac{1}{y_j^2+1}\right]$ where $-\infty < y_j < \infty$, then $Y = \sum_{j=1}^{N} y_j/N$ also has the same Lorentzian distribution, not the Gaussian.

(ii) (5 points) This seems to be in contradiction to the central limit theorem. Is the central limit theorem valid for any probability distribution functions? (Hint: What do the first moments look like?).