Statistical Mechanics WS 2013/14 Sheet 10

Please hand in your solutions (in pairs) before the Monday lecture.

Problem 1 : Quantum Statistics I (5 points)

Enumerate the number of ways to put 3 particles into 3 single-particle states assuming that the particles are

a) bosons

b) fermions

c) distinguishable (classical)

Problem 2 : Quantum Statistics II (15 points)

a) (5 points) Write down the single partition function Z_1 for a monatomic ideal gas, in terms of the total volume V and thermal wavelength (or so-called the quantum wavelength) $\lambda = \sqrt{\frac{h^2}{2\pi m k_B T}}$. [Tip: Remember the partition function is always "dimensionless".]

b) (5 points) When $Z_1 \gg N$, i.e., the number of accessible singe particle states is a way larger than the total number of particles in a system, we can use the classical statistics. Re-express this condition in terms of the density $\rho = N/V$ and λ.

c) (5 points) When the above condition breaks down, i.e., $Z_1 \approx N$, we must consider the quantum statistics. Describe some examples (depending on variables) of "quantum situations" based on this condition. [Hint: There are three variables; density, temperature and mass.]

Problem 3 : The Distribution Functions (20 points)

a) (7 points) For non-interacting fermions, express the probability $P_{\mathrm{FD}}(n, \langle n \rangle_{\mathrm{FD}})$ for finding n particles in a certain energy state per particle, in terms of n and $\langle n \rangle_{\mathrm{FD}}$.

b) (7 points) For non-interacting bosons, express the probability $P_{\mathrm{BE}}(n, \langle n \rangle_{\mathrm{BE}})$ for finding n particles in a certain energy state per particle, in terms of n and $\langle n \rangle_{\mathrm{BE}}$.

c) (6 points) For N ideal gas particles, express the probability $P_{\mathrm{MB}}(N, \langle n \rangle_{\mathrm{MB}})$ for finding any single particle in a certain energy state, in terms of N and $\langle n \rangle_{\mathrm{MB}}$.

Problem 4 : Electrons in a metal (20 points)

- Each atom in a chunk of copper contributes one conduction electron. Look up the density and atomic mass of copper and calculate
  a) the density n of the conduction electrons,
  b) the Fermi energy,
  c) the thermal wavelength λ of the electrons at room temperature,
  d) and the quantity $n\lambda^3$.

  Can the electrons be described classically?

- For a system of fermions at room temperature, compute the probability of a single-particle state being occupied if its energy is
  a) 1 eV less than the chemical potential μ,
  b) 0.01 eV less than μ,
  c) equal to μ,
  d) 0.01 eV greater than μ,
  e) and 1 eV greater than μ.

Problem 5 : Virial expansion for the ideal quantum gas (20 points)

Consider a gas of non-interacting free particles that obeys quantum statistics. The average occupation number for a state with momentum $\vec{p} = \hbar\vec{k}$ is given by

$$n_{\vec{k}} = \frac{1}{z^{-1}e^{\beta\epsilon_{\vec{k}}} \pm 1} \tag{1}$$

where $z = e^{\beta\mu}$ is the fugacity, $\beta = \frac{1}{kT}$, $\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$ is the single particle energy and $\pm$ stands for $+$ (fermions) and $-$ (bosons). The total number of electrons, N, and the pressure, P, they exert can be written as

$$N = \frac{1}{V}\sum_{\vec{k}} n_{\vec{k}} \tag{2}$$

$$P = \frac{1}{V}\frac{2}{3}\sum_{\vec{k}} \epsilon_{\vec{k}} n_{\vec{k}} \tag{3}$$

a) Approximate the sums by integrals ($\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int dk^3$) and make the variable transformation $\beta\epsilon_{\vec{k}} = x^2$ to show that:

$$\rho\lambda^3 = \frac{4}{\sqrt{\pi}} z \int_0^\infty \frac{x^2}{e^{x^2} \pm z} dx \tag{4}$$

$$\frac{\lambda^3 P}{kT} = \frac{8}{3\sqrt{\pi}} z \int_0^\infty \frac{x^4}{e^{x^2} \pm z} dx \tag{5}$$

where λ is the thermal wavelength and ρ is the density.

b) Expand the integrals around $z = 0$ up to order $O(z^3)$ and solve them to express the density and pressure as a polynomial in the fugacity z.

c) Invert the series $\rho(z)$ to find $z(\rho)$ and substitute the result into the expression for the pressure to obtain $P(z(\rho)) = P(\rho)$. This is the virial expansion.

d) Plot $\frac{P}{kT\rho}$ against $\rho\lambda^3$ for the ideal gas, the ideal fermi gas and the ideal boson gas. In a few words, compare your results with the virial expansion for an interacting ideal gas (cf. problem 4 on sheet 7). What are the reasons for the corrections in the quantum and in the classical case?

Hint: You are encouraged to use *Mathematica* to solve this problem. The functions `Series`, `InverseSeries`, `Integrate` and `NIntegrate` can perform the Taylor expansions and integrations for you.