Sheet 11

Please hand in your solutions before the Wednesday lecture Jan 28 at 10:15.

Problem 1 : Boltzmann, Fermi-Dirac and Bose-Einstein statistics (10 points)

Consider a two-particles system with two energy levels and energies $\varepsilon_0 = 0$, and $\varepsilon_1 = \varepsilon$. Sketch all possible microstates and write down the canonical partition functions for the two particles being

a) (4 points) classical particles, distinguishable and indistinguishable, respectively

b) (2 points) fermions with same spin $s_z = \frac{1}{2}$

c) (2 points) fermions with different spin

d) (2 points) bosons.

Problem 2 : Occupation numbers and fluctuations (20 points)

Consider a quantum system whose energy state occupied by n particles is $n\epsilon$. The chemical potential in the system is μ.

a) (4 points) Show that the mean number of particles $\langle n \rangle$ can be expressed by $\langle n \rangle = -\frac{1}{\Xi}\frac{\partial \Xi}{\partial x}$, where Ξ is the partition function, and $x \equiv (\epsilon - \mu)/k_B T$.

b) (4 points) Show that the mean-squared number of particles $\langle n^2 \rangle$ can be expressed by $\langle n^2 \rangle = -\frac{1}{\Xi}\frac{\partial \langle n \rangle \Xi}{\partial x}$.

c) Now consider the quantum system where n goes up to 2 ($0 \leq n \leq 2$).

   1) (6 points) Find the mean energy $\langle n\epsilon \rangle$ in terms of $k_B T$, x and μ, and sketch a plot of $\langle n\epsilon \rangle$ in units of $k_B T$ as function of $-1 < x < 5$ at $\mu/k_B T = 0.1$.

   2) (6 points) Find the number fluctuation $\Delta n^2 \equiv \langle n^2 \rangle - \langle n \rangle^2$ in terms of x, and sketch a plot of Δn^2 as function of $-10 < x < 10$.

Problem 3 : Quantum corrections to the classical limit (20 points)

Quantum statistics becomes important when the thermal wavelength λ is close to the average distance between particles $v^{\frac{1}{3}} = \left(\frac{V}{N}\right)^{\frac{1}{3}}$:

$$\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{\frac{1}{2}} = v^{\frac{1}{3}}. \quad (1)$$

a) (5 points) Find the threshold temperature for the following systems (N=1 mol=6.22×10^{23}):

	m/amu	$V/10^{-4}\mathrm{m}^3$
Argon	40	8
Neon	20	5
Helium	4	4
Nitrogen	28	0.5
Water	18	10
Electron	0.00054	10

(1 amu=1.66×10^{-27} kg, h=6.63×10^{-34} J · s, $k_B = 1.38 \times 10^{-23}$ J · K^{-1})

b) (5 points) Find the threshold volume for the following systems (N=1 mol):

	m/amu	T/K
Argon	40	100
Neon	20	100
Helium	4	10
Nitrogen	28	200
Water	18	300
Electron	0.00054	300

c) (5 points) Quantum corrections can be introduced by a quantum mechanical virial expansion. How large is the quantum correction at the treshold temperatures/volumes from task a) and b) (use again N=1 mol)?

$$\frac{pV}{k_B T} = N\left(1 + B_{qm}\frac{N}{V}\right), \tag{2}$$

$$B_{qm} = \pm\frac{\lambda^3}{(2s+1)\cdot 2^{\frac{5}{2}}} = \begin{cases} \frac{\lambda^3}{2^{\frac{7}{2}}} & \text{for } s = \frac{1}{2} \\ -\frac{\lambda^3}{2^{\frac{5}{2}}} & \text{for } s = 0 \end{cases}. \tag{3}$$

d) (5 points) Compare the quantum corrections to the classical corrections for interacting gases $B_{\text{classic}}\frac{N}{V}$ according to

$$\frac{pV}{k_B T} = N\left(1 + B_{\text{classic}}\frac{N}{V}\right), \tag{4}$$

with $B_{\text{classic}} = b - \frac{a}{kT}$ for

	$a/(101325\ \text{J m}^3/\text{kmol}^2)$	$b/(\text{m}^3/\text{kmol})$
Argon	1.355	0.0320
Neon	0.2135	0.0171
Helium	0.0346	0.0237
Nitrogen	1.408	0.0391
Water	5.536	0.0305