Sheet 6

Please hand in your solutions before the Monday lecture at 10:15.

Problem 1 : Maxwell-Boltzmann distribution (30 points)

a) Using the Maxwell-Boltzmann distribution of velocities for an ideal monoatomic gas

$$w(v)\,dv = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} 4\pi v^{\mathbf{2}} \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

show that

1) (2 points) the average velocity is

$$\langle v\rangle = \left(\frac{8k_B T}{\pi m}\right)^{\frac{1}{2}}$$

2) (2 points) the average squared velocity is

$$\left\langle v^2\right\rangle = \frac{3k_B T}{m}$$

3) (2 points) the most probable velocity is

$$v_{max} = \sqrt{\frac{2k_B T}{m}}$$

4) (1 points) the (kinetic) energy per atom is

$$E = \frac{3}{2}k_B T$$

b) For Argon (m_{Ar}=39.948g/mol) at temperatures $T = 100, 150, \ldots, 500K$

1) (2 points) Compute and plot the probability density of velocities.
2) (2 points) Calculate the average velocity.
3) (2 points) Calculate the average squared velocity.
4) (2 points) Calculate the most probable velocity.
5) (2 points) Calculate the kinetic energy per atom.

c) Consider an ideal gas in one dimension x.

1) (4 points) Derive the Maxwell-Boltzmann distribution of velocities $w(v_x)$ from the canonical ensemble average of the velocity.
2) (2 points) Show that the average velocity in one dimension is $\langle v_x\rangle = 0$.
3) (2 points) Find the average squared velocity in one dimension.
4) (5 points) Redo tasks b)1)-b)5) in one dimension.

Problem 2 : n-leg animal (20 points)

Consider an animal called "Alice" with n identical legs in total. In the "Alice-world" system of volume V and temperature T, N Alices (indistinguishable and non-interacting) can enjoy their entropy by floating around in the air, which leads to the ideal gas partition function $\mathcal{Z}_N^{\text{ideal}} = V^N/(N!\lambda^{3N})$ where $\lambda = h/\sqrt{2\pi m k_B T}$ is the thermal wavelength. Or, they can land on the Wonderland which is full of nutrients. For landing they can use at least one of n legs, thereby each of Alices can take nutrients via their legs, which leads to the energy decrement $U = -\epsilon$ per a leg. In this case, the only one remaining degree of freedom is number of legs out of n legs.

a) (5 points) Calculate the single partition function $\mathcal{Z}_1$ and the total partition function $\mathcal{Z}_N$.

b) (10 points) Calculate the Helmholtz free energy $\mathcal{A} = -k_B T \ln \mathcal{Z}_N$ for large N. Plot $\mathcal{A}$ as a function of $500 < N < 2000$ using $n = 3$, $k_B T = 4$ nm pN, $\epsilon = 1$ nm pN, and $V = 100\lambda^3$.

c) (5 points) Calculate the chemical potential μ.