Problem sheet 14

Please hand in your solutions before the lecture on Wednesday, 3th of February.

Problem 1 - Bose-Einstein condensation

(a) For low temperature $T < T_c$ the pressure of an ideal Bose-Einstein gas is

$$P = k_B T \, \frac{g_{5/2}(\xi)}{\lambda^3}, \tag{1}$$

where λ is the thermal wave length $\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$, $\xi = \exp(\beta\mu)$, and g is the Bose function. Use the Clausius-Clapeyron equation for the phase equilibrium between particles in the condensate and particles in the gas and find the condensation enthalpy ("heat of condensation"). Hint: the condensate has (essentially) no volume. *(4 points)*

(b) Heat Capacity of a d-dimentional ($d > 2$) non-interacting bosons gas in $T < T_c$ and $z = 1$ is:

$$C(T) = \frac{d}{2}\Big(\frac{d}{2}+1\Big)\frac{V}{\lambda^d} k_B \zeta_{\frac{d}{2}+1} \qquad T_c = \frac{h^2}{2mk_B}\Big(\frac{n}{\zeta_{\frac{d}{2}}}\Big)^{2/d} \tag{2}$$

b-1) Sketch the heat capacity $C(T)$ for all temperatures. *(2 points)*

b-2) Find the ratio of the maximum heat capacity to its classical limit, $C_{max}(T)/C(T \to \infty)$, and evaluate it for $d = 3$. *(4 points)*

b-3) How does the above ratio $C_{max}(T)/C(T \to \infty)$ behave for $d = 2$? Explain. *(3 points)*

Problem 3 - Photon gas

The partition function of a Photon gas in logarithmic form is

$$\ln Z = \frac{V}{\hbar^3 c^3 \beta^3}\frac{\pi^2}{45}. \tag{3}$$

a) Show that the photon pressure is

$$\mathrm{P} = \frac{1}{3}\frac{E}{V} = \frac{4}{3}\frac{\sigma}{c}T^4,$$

where c is the speed of light and σ is the Stefan-Boltzmann constant $\sigma = \frac{k_B^4 \pi^2}{60\hbar^3 c^2}$. *(3 points)*

b) Show that the entropy of a photon gas is *(3 points)*

$$S = \frac{16}{3}\sigma V T^3.$$

c) Use the Gibbs-Duhem relation

$$\langle N \rangle \mu = E - TS + \mathrm{PV},$$

to show that the chemical potential must be zero if $\langle N \rangle \neq 0$ which is the case for photons . *(2 points)*

Problem 3 - Debye solid

The average energy U of a crystal of N particles in three dimension can be approximated as

$$U = \frac{9Nk_BT^4}{T_D^3}\int_0^{T_D/T}\frac{x^3}{e^x-1}dx, \tag{4}$$

where T_D is the Debye temperature.

a) The particles in the crystal can be modelled as N harmonic oscillators in three dimension. Using the equipartition theorem find the average energy U. *(2 points)*

b) Using Eq. (4), show that U for $T \gg T_D$ reduces to the classical energy of an ideal gas. *(3 points)*

c) Using Eq. (4), show that U for $T \ll T_D$ leads to $U = 3\pi^4 Nk_BT^4/(5T_D^3)$. *(2 points)*

d) Find the heat capacity C_V for $T \ll T_D$. *(2 points)*