Luttinger-liquid universality in the time evolution after an interaction quench

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We provide evidence that the relaxation dynamics of one-dimensional, metallic Fermi systems resulting out of an abrupt amplitude change of the two-particle interaction has aspects which are universal in the Luttinger liquid sense: the leading long-time behavior of certain observables is described by universal functions of the equilibrium Luttinger liquid parameter and the renormalized velocity. We analytically derive those functions for the Tomonaga–Luttinger model and verify our hypothesis of universality by considering spinless lattice fermions within the framework of the density-matrix renormalization group.

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The equilibrium low-energy physics of a large class of one-dimensional ($1d$), correlated, metallic Fermi systems is described by the Luttinger liquid (LL) phenomenology [1,2]. The Tomonaga–Luttinger (TL) model is the effective low-energy fixed point model of the LL universality class and thus, plays the same role as the free Fermi gas in Fermi liquid theory. This universality relies on the infrared renormalization group (RG) irrelevance of contributions such as the momentum dependence of the two-particle interaction [3] or the curvature of the single-particle dispersion [4], which are present in microscopic models but ignored in the TL model. For a model falling into the LL universality class, it is not necessary to explicitly compute thermodynamic observables and correlation functions if one is interested in the low-energy limit. One only needs to determine two numbers—the LL parameter K and the renormalized velocity v of the excitations—which fully characterize the low-energy physics of a spinless LL (on which we focus). Those depend on the band structure and filling as well as the amplitude and range of the two-particle interaction of the microscopic model at hand and can be extracted from the ground-state energy [5] or 'simple' response functions [6]. Thereafter, correlation functions at long length scales or thermodynamic quantities at low energies can be obtained by plugging in K and v into analytic expressions derived within the exactly solvable TL model [1,2].

The recent progress in experimentally controlling isolated many-body states, in particular in cold atomic gases [7], led to numerous theoretical studies on the dynamics of closed quantum systems resulting out of an abrupt change of the amplitude U of the two-particle interaction [8]. One assumes that the system is prepared in a canonical thermal state or, at temperature $T = 0$ on which we focus, the ground state of an initial Hamiltonian H_i with a given U_i; often $U_i = 0$ is considered. At time $t = 0$, the interaction is quenched to U_f and the time evolution is performed with the final Hamiltonian H_f. Fundamental questions discussed are [8]: (i) Do some observables become stationary at large times? (ii) How can they be classified (locality)? (iii) Is it possible to compute their steady-state expectation values using an appropriate density-matrix ρ_{st}? It was conjectured that in models with many integrals of motion, e.g., those which are solvable by Bethe ansatz ('integrable') [9–11], ρ_{st} is not of thermal but rather of generalized ('Gibbs') canonical form [12]. This has been confirmed for models which can be mapped to effective noninteracting ones [12–18], in particular the TL model [19] and a variant of the latter [20], but a general proof is lacking. For generic models, one generally expects ρ_{st} to be thermal.

Here, we address questions about the relaxation dynamics towards the steady state. Considering several models which in equilibrium fall into the LL class we ask: (1) Can the time evolution be characterized as universal in the LL sense? As model-dependent high energy processes matter at short to intermediate times, one can only expect to find LL universality on large time scales—but even then it is not obvious whether the notion of RG irrelevance can be transferred from equilibrium to the nonequilibrium dynamics [21–23]. (2) If LL universality is found, does it hold independently of the number of integrals of motion and thus, independently of the expected nature of the steady state (generalized canonical versus thermal)? To answer those questions we proceed in steps. We compute the time evolution of the Z factor jump of the momentum distribution function $n(k, t)$ at the Fermi momentum k_F and the kinetic energy per length $e_{kin}(t)$ for the spinless TL model with arbitrary momentum dependent interactions $g_{2/4}(k)$ using bosonization [1,2]. $e_{kin}(t)$ is defined as the expectation value of the initial Hamiltonian H_i and thus, describes how excitations die out. The quench is performed out of the noninteracting ground state. We analytically show that for any continuous $g_{2/4}(k)$ the long-time dynamics are given by [24]

$$Z(t) \sim t^{-\gamma_{st}(K)}, \tag{1a}$$

$$|de_{kin}(t)/dt| \sim \epsilon(K, v) t^{-3}. \tag{1b}$$

The decay of $Z(t)$ is governed by a universal exponent $\gamma_{st}(K)$ that depends on the equilibrium LL parameter K only; $e_{kin}(t)$ features an asymptotic power law with an interaction-independent exponent but universal prefactor $\epsilon(K, v)$ determined by K as well as by the renormalized velocity v. In equilibrium, this is the characteristic K- and v-dependence of correlation functions or of thermodynamic quantities, which *a posteriori* motivates to consider $Z(t)$ and $e_{kin}(t)$ as representative examples. The notion of LL universality can now be defined in analogy to equilibrium: the quench dynamics is universal if Eqs. (1a) and (1b) describe the long-time relaxation for any model falling into the equilibrium LL universality class if the corresponding values for K and v are plugged in. To investigate this, we compute $Z(t)$ for a $1d$ lattice of spinless fermions with nearest-neighbor hopping and interaction Δ [25] as well as an extension of the latter including a next-to-nearest-neighbor interaction Δ_2. We use the numerical time-dependent density-matrix RG (DMRG) [26–28]. The model with $\Delta_2 = 0$ has many conserved quantities, is Bethe ansatz integrable, and thus, K as well as v are known analytically [5,29]. The Δ-Δ_2–model, however, is believed to be not exactly solvable. For $\Delta_2 > 0$, we extract K and v from equilibrium quantities (e.g., the small momentum density response function) using DMRG [6,30,31]. Our data for the Z factor agree with Eq. (1a) for any interaction strength, filling factor, and irrespective of the integrability of the model. The results for $e_{kin}(t)$ are consistent with Eq. (1b), but on the time scales accessible by DMRG, the asymptotic behavior is still masked by oscillatory terms of higher order in t^{-1}. To unambiguously determine the prefactor of the t^{-3} decay of the energy, we resort to a numerical trick. Instead of performing the time evolution with $\exp(-iH_f t)$, we apply the imaginary time analogue $\exp(-H_f\tau)$. In this case, the total energy per length $e(\tau)$—which is no longer conserved—is the natural observable. For the TL model, we show that the asymptotics is completely analogous to Eq. (1b) with $t \to \tau$ and $\epsilon(K, v)$ replaced by a different function $\epsilon_{it}(K, v)$. For the lattice model, the τ^{-3} decay manifests over several orders of magnitude, and the prefactor agrees with the TL prediction.

This altogether provides evidence that questions (1) and (2) can be answered by 'yes'. We conjecture that the universality of the quench dynamics also holds for other models falling into the equilibrium LL class.

The Tomonaga–Luttinger model.—After bosonizing [1,2] the density of left and right moving fermions with a linear dispersion, the Hamiltonian of the TL model is quadratic in operators $b_n^{(\dagger)}$, which obey bosonic commutation relations:

$$H = \sum_{n>0}\left[k_n\left(v_F + \frac{g_4(k_n)}{2\pi}\right)(b_n^\dagger b_n + b_{-n}^\dagger b_{-n}) + k_n\frac{g_2(k_n)}{2\pi}(b_n^\dagger b_{-n}^\dagger + b_{-n}b_n)\right], \tag{2}$$

where $k_n = 2\pi n/L$, $n \in \mathbb{Z}$, L denotes the chain length, and v_F is the Fermi velocity. The two coupling functions (potentials) $g_{2/4}$ determine the strength of the scattering of fermions on different branches (g_2) and the same branch (g_4). Usually the k-dependence of $g_{2/4}$ is neglected and integrals are regularized in the ultraviolet through an *ad hoc* procedure [1,2]. As the momentum dependence is RG irrelevant, this is justified in equilibrium if all energy scales are sent to zero [3]. For the quench dynamics—even at asymptotic times—it is, however, not clear if the same reasoning holds and we thus, keep the full k-dependence and consider coupling functions. In fact, it was recently shown that the momentum dependence indeed affects the long-time dynamics of certain observables [22]. For the system to be a LL in equilibrium, we require that $0 < g_{2/4}(0) < \infty$ (repulsive interactions) and that $g_{2/4}(k)$ decay on a scale k_c. The Hamiltonian of Eq. (2) can be diagonalized to $H = \sum_{n\neq 0}\omega(k_n)\alpha_n^\dagger\alpha_n + E_{gs}$ by introducing new modes $\alpha_n = c(k_n)b_n + s(k_n)b_{-n}^\dagger$ with

$$s^2(k) = \frac{1}{2}\left[\frac{1+\hat{g}_4(k)}{W(k)} - 1\right] = c^2(k) - 1,$$

$$\omega(k) = v_F|k|W(k) = v_F|k|\sqrt{(1+\hat{g}_4(k))^2 - \hat{g}_2^2(k)}, \tag{3}$$

where $\hat{g}_{2/4} = g_{2/4}/(2\pi v_F)$, and E_{gs} denoting the ground state energy. The LL parameter and the renormalized velocity read

$$K = \sqrt{\frac{1+\hat{g}_4(0) - \hat{g}_2(0)}{1+\hat{g}_4(0)+\hat{g}_2(0)}}, \qquad v = v_F W(0). \tag{4}$$

As our initial state, we take the noninteracting ground state $|E_{gs}^0\rangle$ which is given by the vacuum $|\text{vac}(b)\rangle$ with respect to the b_n. Expectation values of the time-evolved state $|\Psi(t)\rangle = \exp(-iHt)|E_{gs}^0\rangle$ can be computed straightforwardly using the simple time dependence of the eigenmode operators $\alpha_n^{(\dagger)}$ and their linear dependence on the $b_n^{(\dagger)}$ [22].

After bosonizing the fermionic field operator [1,2], the Z factor $Z(t) = \lim_{k\nearrow k_F} n(k,t) - \lim_{k\searrow k_F} n(k,t)$ is easily obtained (taking $L \to \infty$) [19,22,32]:

$$Z(t) = \exp\left\{-\int_0^\infty dk\frac{4s^2(k)c^2(k)}{k}(1 - \cos[2\omega(k)t])\right\}.$$

Independent of the form of $g_{2/4}(k)$ (even for potentials with a discontinuous jump to zero at k_c), the large-time behavior is given by Eq. (1a) with $\gamma_{st} = (K^2 + K^{-2} - 2)/4$; it manifests on the (nonuniversal) scale $(v_F k_c)^{-1}$. Figure 1(a) shows $Z(t)$ obtained by numerically performing the integral for a simple box shaped potential $\hat{g}_2(k) = \hat{g}_4(k) = g\Theta(k_c - |k|)/2$ of varying amplitude g. The asymptotic power law is modulated by oscillations which decay faster than $t^{-\gamma_{st}}$.

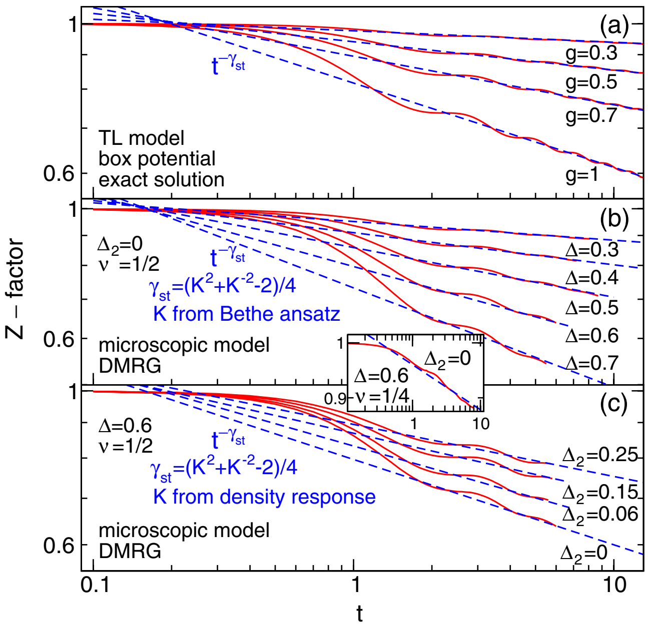

FIG. 1 (color online). Time evolution of the Z factor out of the noninteracting ground state of a $1d$ metallic Fermi system after switching on two-particle terms at time $t = 0$. Dashed lines show the universal asymptotic power law $t^{-\gamma_{\rm st}(K)}$ with an exponent determined by the equilibrium LL parameter K. (a) TL model. The plots displays boxlike two-particle interactions $g(k)$ of strength $g = g(0)$; the asymptotics are universal for any $g(k)$. Time is given in units of $(v_F k_c)^{-1}$. (b, c, Inset) Spinless lattice fermions of Eq. (6) at filling ν featuring nearest (Δ) and next-nearest (Δ_2) neighbor interactions.

The kinetic energy per length $e_{\rm kin}(t)$ reads ($L \to \infty$)

$$e_{\rm kin}(t) = \frac{v_F}{2\pi}\int_0^\infty dk k 4 s^2(k) c^2(k)\{1 - \cos[2\omega(k)t]\}. \qquad (5)$$

The steady-state value is obtained by dropping the oscillatory term which averages out for $t \to \infty$. For continuous coupling functions $g_{2/4}(k)$ of range k_c, asymptotic analysis yields Eq. (1b) as the leading term in the long-time limit [33]; the coefficient is given by $\epsilon(K, v) = \gamma_{\rm st}(K) v_F/(4\pi v^2)$. Figure 2(a) shows the derivative of $e_{\rm kin}$ for $\hat{g}_2(k) = \hat{g}_4(k) = g(k)$, a Gaussian potential $g(k) = g\exp(-[k/k_c]^2/2)/2$ as well as a quartic potential $g(k) = g/(1 + [k/k_c]^4)/2$ and varying interaction strengths. As either $g(0)$ or the lowest nonvanishing Taylor expansion order of $g(k) - g(0)$ increases, the amplitude of an oscillatory term which decays faster than the leading one becomes stronger. The (nonuniversal) scale on which the asymptotic t^{-3}-behavior dominates, thus, heavily depends on the strength and type of potential at hand [compare the inset and the main part of Fig. 2(a)].

Microscopic lattice model.—As a next step, we provide evidence that Eqs. (1a) and (1b) describe the long-time relaxation dynamics of any model which in equilibrium falls into the LL universality class. To this end, we consider spinless lattice fermions,

$$H = \sum_j \left[\frac{1}{2} c_j^\dagger c_{j+1} + \text{H.c.} + \Delta n_j n_{j+1} + \Delta_2 n_j n_{j+2}\right], \qquad (6)$$

with $n_j = c_j^\dagger c_j - 1/2$. We study the quench dynamics using an infinite-system DMRG algorithm [26,34]. We determine $|E^0_{\rm gs}\rangle$ by applying an imaginary time evolution, $\exp(-\tau H|_{\Delta=\Delta_2=0})$, to a random initial matrix product state with a fixed matrix dimension χ until the energy has converged to typically 8–10 relative digits. Operators, $\exp(\sim H)$, are factorized by a second or fourth order Trotter decomposition. Thereafter, we compute the real time evolution $|\Psi(t)\rangle = \exp(-itH)|E^0_{\rm gs}\rangle$ in presence of the two-particle terms Δ and Δ_2. χ is dynamically increased in order to maintain a fixed discarded weight. We carefully ensure that the latter is chosen small enough (and that the initial χ is large enough) to obtain numerically exact results.

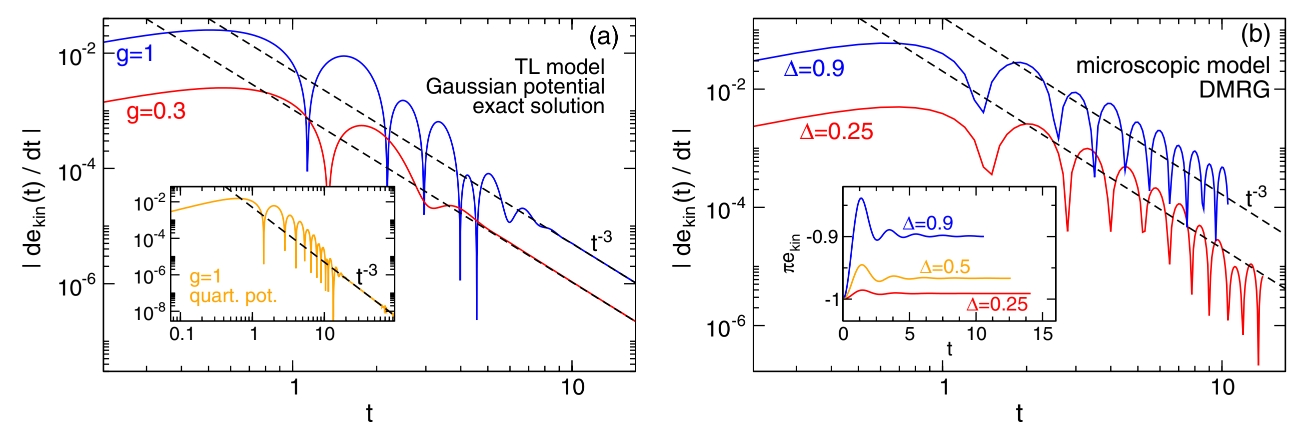

FIG. 2 (color online). Time evolution of the kinetic energy per length $de_{\rm kin}/dt$. (a) TL model for different (Gaussian and quartic) two-particle potentials $g(k)$. For any continous $g(k)$, $de_{\rm kin}/dt$ asymptotically falls off as $\epsilon(K, v)/t^3$ with a universal prefactor. $e_{\rm kin}$ and t are given in units of $v_F k_c^2$ and $(v_F k_c)^{-1}$, respectively. (b) Spinless lattice fermions. Solid lines show DMRG data; dashed lines display t^{-3} power laws, where the ratio of prefactors is chosen according to the TL prediction. Inset: DMRG data before taking the t derivative.

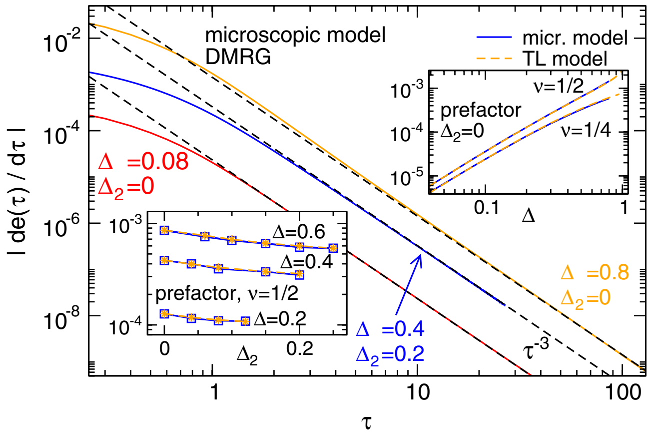

FIG. 3 (color online). Imaginary time evolution $\exp(-\tau H)|E^0_{\rm gs}\rangle \to^{\tau\to\infty} |E_{\rm gs}\rangle$ towards the interacting ground state $|E_{\rm gs}\rangle$. The main part shows DMRG data for the τ derivative of the total energy in the lattice model. The τ^{-3} decay predicted by bosonization manifests over several orders of magnitude. The prefactor agrees with the TL formula $\epsilon_{\rm it}(K, v)$ for all parameters (this is illustrated in the insets). The imaginary-time energy dynamics are, thus, universal in the LL sense.

The time evolution of the momentum distribution function $n(k, t) = \sum_j e^{ikj}\langle\Psi(t)|c_j^\dagger c_0|\Psi(t)\rangle$ and the corresponding Z factor can be computed straightforwardly; the j sum is carried out up to ~10000 sites. Results for $Z(t)$ are shown in Figs. 1(b) and 1(c). Its long-time asymptotics indeed shows a power law decay $t^{-\gamma_{\rm st}(K)}$, and the exponent agrees to the one predicted by the TL formula (dashed lines) if the latter is evaluated for K, corresponding to the microscopic parameters under consideration. We take K from the Bethe ansatz ($\Delta_2 = 0$) [5,29] or the equilibrium density response ($\Delta_2 > 0$) [6,30]. The agreement with the TL model result holds for any interaction strength [31], for any filling factor ν, and irrespective of the integrability of the model. This strongly indicates that the asymptotic dynamics of the Z factor is indeed universal in the LL sense.

The time derivative of the kinetic energy (the expectation value of $H|_{\Delta=\Delta_2=0}$) per length (site) is shown in Fig. 2(b). Its magnitude decays as t^{-3}, and the ratio between prefactors (where the factor $v_{\rm F}$ drops out) at different interaction strengths is consistent with the TL formula (see the dashed lines). However, oscillations have not died out completely, and a pure power law cannot be identified unambigously. To further support that this is merely because the time scales reachable in our DMRG calculation are too small—remind that for the TL model the scale where $de_{\rm kin}/dt$ is governed by Eq. (1b) strongly depends on the strength and type of the potential in contrast to $Z(t)$ where it is always $(v_{\rm F}k_c)^{-1}$–and that the energy relaxation is indeed universal, we consider an imaginary time evolution, $|\Psi(\tau)\rangle = \exp(-H\tau)|E^0_{\rm gs}\rangle/\langle E^0_{\rm gs}|\exp(-2H\tau)|E^0_{\rm gs}\rangle^{1/2}$. For $\tau \to \infty$, $|\Psi(\tau)\rangle$ approaches the ground state $|E_{\rm gs}\rangle$ of the interacting Hamiltonian. The total energy is the natural observable to compute in this academic scenario. Its asymptotic behavior within the TL model is completely analogous to Eq. (1b) with $t \to \tau$ and $\epsilon(K, v) \to \epsilon_{\rm it}(K, v) = {\rm Li}_2([K + K^{-1} - 2]/[K + K^{-1} + 2])/(8\pi v)$, where ${\rm Li}_2$ denotes the dilogarithm [35]. For the lattice model, one can easily access large imaginary times using DMRG, and the τ^{-3} decay manifests over several orders of magnitude. This is illustrated in Fig. 3. The prefactor (shown in the insets) agrees with $\epsilon_{\rm it}(K, v)$ (the latter depends on K and v only; thus, one does not need to consider ratios) for all interactions and fillings. The dynamics of the total energy at large τ is universal.

Conclusion.—We have obtained exact expressions for the time evolution of the Z factor and of the kinetic energy after an interaction quench within the TL model. For any continuous two-particle potential, their long-time asymptotes $Z \sim t^{-\gamma_{\rm st}}$, $de_{\rm kin}/dt \sim \epsilon(K, v)/t^3$ are universal functions of the LL parameter K and the renormalized velocity v. We studied a similar scenario for spinless lattice fermions using DMRG; for large times, $Z(t)$ and $e_{\rm kin}(t)$ are described by the above expressions. This provides evidence that the relaxation dynamics after an interaction quench within any model that falls into the equilibrium LL universality class has aspects which are universal in the LL sense. Universality holds independent of the supposed nature of the steady state (thermal versus generalized Gibbs) and it would be very interesting to understand this in more detail.

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