VL Mi 14-16 Uhr : Prof. Dr. Kathy Lüdge
UE Mi 16-18 Uhr (2wöchig)

1. Homework - Nonlinear dynamics

Need to be handed in: Wed 28.10.(before the lecture) You can build groups of up to 3 students.

Task 1 (10 points): Maxwell-Bloch-Equations
The Maxwell Bloch equations describe the dynamics of a laser. Below they are given in their dimensionless form

$$\dot{E} = \kappa(P - E),$$
$$\dot{P} = \gamma_1(E\,D - P),$$
$$\dot{D} = \gamma_2(\mathrm{J} + 1 - D - E\,P).$$

Here, E is the electric field amplitude of the laser mode, P is the microscopic polarisation of the medium and D labels the inversion of the electronic levels. The parameter $\kappa > 0$ is the photon loss rate, while $\gamma_1 > 0$ and $\gamma_2 > 0$ are the decay rates of the polarisation and the inversion, respectively. The parameter J lables the excess pump rate (compared to the threshold value) and can be positiv zero or negativ. These equations are equivalent to the Lorenz equations and can show complicated dynamics, including chaos.

1. Find the fixed foints and characterize them (stable/unstable and saddle/node/focus).

2. For most laser-systems there is a time scale separation, i.e. $\gamma_1, \gamma_2 \gg \kappa$. Thus the equations can be simlified and P and D can be adiabaticallyt eliminated. Use the static relations resulting from $P^{\cdot} \approx 0$ and $D^{\cdot} \approx 0$ and derive a first order diffenertial equation for the field E .

3. Find the fixed points E^* for this equation and characterize the stability. Draw a bifurcation diagram (E^* as a function of J).

Task 2(10 points): *Van der Pol Oszillator*
In the 1930th Van der Pol started to investigate an oszillator with an additional nonlinear damping term. The equation reads:

$$x^{\cdot\cdot} + \kappa(x^2 - a)x^{\cdot} + {\omega_0}^2 x = 0 \; (\kappa \geq 0).$$

For $a > 0$ the term acts like a usual fricion term, however for small amplitudes x it can change sign and thus lead to anharmonic oscillations.

1. Write the equation as a first order ODE system by defining a second variable $y = x^{\cdot}$ and find the fixed points of the dynamics and their stability.

2. Solve the equation numerically using the parameter $\kappa = 1$, $\omega_0 = 1$ and 1000 and different values for a (between -1 and 1). Draw a bifurcation diagram as a function of a . For that you have to find the maxima and minima of the time trace (after some time has passed to avoid transients).
   Plot the phase portait in the (x, y)-plane for $a = 0.1$, $a = 1$, $a = 2$. Interpret the results.