VL: Mi 14-16 Uhr Prof. Dr. Kathy Lüdge
UE: Mi 16-18 Uhr (every 2nd week)
http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-luedge/teaching/

2. Homework - Nonlinear dynamics

Need to be handed in: Wed 11.11.(before the lecture) You can build groups of up to 3 students.

Task 3 (10 points): *Homoclinic orbit in the Lorenz system*

A homoclinic orbit is a connection from a fixed point to itself that can usually only be found numerically. In this exercise you are supposed to find such a homoclinic connection for a fixed point of the Lorenz system.
The Lorenz system is given by

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - \beta z,$$

where we chose $\sigma = 10$, $\beta = 8/3$ and $1 < \rho < 24$. In this parameter range there are three fixed points, which we call C_0, C_+ and C_- . The last two fixed points are stable foxi:

$$C_0 := (0, 0, 0),$$
$$C_\pm := (\pm\xi, \pm\xi, \rho - 1) \text{ mit } \xi = \sqrt{\beta(\rho - 1)}.$$

1. Show that C_0 and $C_\pm$ are indeed fixed points of the system. Further, please show that for $\rho > 1$ C_0 is a saddle with two stable and one unstable directions . Determine the eigenvector ξ_0 that corresponds to the unstable direction of C_0 (this vector is a vector tangent to the unstable manifold of C_0).

Now you are ready to determine the homoclinic orbit that, however, only exists for one specific value $\rho = \rho_*$.

2. Start your simulation (for one value of ρ) with initial condition at the fixed point C_0 with a small deviation in the direction of ξ_0 . For $\rho < \rho_*$ your dynamics will end in one of the fixed points $C_\pm$ while for $\rho > \rho_*$ it will end in the other one.

   Implement an iteration procedure that evaluates the phase portraits in the (x,y) plane in order to determine ρ_* with an accuracy of 4 digits. Please note that your simulation should run at least until $t = 100$ with a small stepsize to yield the desired accuracy.

3. Plot the phase portait of the homoclinic orbit (for that you only need to simulate to $t = 5$).

Task 4 (10 points): *SNIPER or SNIC*

A *SNIPER* (**s**addle-**n**ode **i**nfinite **per**iod) or *SNIC* (saddle node on an infinit cycle) bifurcation can be described by the following set of equations for the radius *r* and the angle ϕ of a rotating motion.

$$\dot{r} = r(1 - r^2),$$
$$\dot{\phi} = b - r\cos\phi.$$

In the following you can reduce the dynamics to $r = 1$ (this is possible because the circle is an invariant manifold of the system).

1. Characterize the fixed points of the reduced system as a function of b, i.e., determine their existence, their value and their stability.
2. Find the analytic solutions $\phi(t)$ of the ODE system by seperating the variables, and distinguish the cases $b < 1$ and $b > 1$. You may need to use the following primitiv integrals:

$$\int \frac{d\phi}{b - \cos\phi} = \frac{2}{\sqrt{b^2 - 1}} \arctan\left[\frac{(b+1)\tan\frac{\phi}{2}}{\sqrt{b^2 - 1}}\right] \qquad \text{for} \quad b > 1,$$
$$\int \frac{d\phi}{b - \cos\phi} = \frac{1}{\sqrt{1 - b^2}} \log\left[\frac{(1+b)\tan\frac{\phi}{2} - \sqrt{1 - b^2}}{(1+b)\tan\frac{\phi}{2} + \sqrt{1 - b^2}}\right] \qquad \text{for} \quad b < 1.$$

3. Plot the resulting time series $x(t) = \cos\phi(t)$ and the phase portrait for $b > 1$ and $b < 1$ for suitable initial conditions.