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Statistical Mechanics: Mathematical Preliminaries

This problem set will not be graded, hence, you don't need to submit it. We will discuss it in the first tutorial to refresh some important mathematical aspects.

1) Volume Elements Under Transformations

The cartesian coordinates x, y, z are transformed into a new set of independent coordinates u, v, w according to

$$x = f_1(u, v, w), \qquad y = f_2(u, v, w), \qquad z = f_3(u, v, w). \tag{1}$$

How does the volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ transform?
Compute the transformed area element $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ explicitly for

$$x = u\cos v, \qquad y = u\sin v. \tag{2}$$

2) The δ Function

For $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, calculate the following integrals

a)
$$\int_{-\infty}^{\infty} \mathrm{d}x\,\delta(x-b)f(ax), \tag{3}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\,\delta(ax-b)f(x). \tag{4}$$

c) Let in addition be $g : \mathbb{R} \to \mathbb{R}$ and let $\{x_1, x_2, \dots, x_k\}$ be the zeros of g, i.e $g(x_1) = g(x_2) = \dots = g(x_k) = 0$, compute
$$\int_{-\infty}^{\infty} \mathrm{d}x\,\delta(g(x-b))f(ax). \tag{5}$$

3) Gaussian Integrals

Prove that, for $a > 0$,

a)
$$\int_{-\infty}^{\infty} \mathrm{d}x\,e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{6}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\,e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a^2}}. \tag{7}$$

4) Fourier Transforms

For $k, a \in \mathbb{R}$ and $a > 0$, calculate the following Fourier transforms and extract the real and imaginary parts of your results.

a)
$$\int_{-\infty}^{\infty} \mathrm{d}x\,e^{-ikx}\delta(x-a), \tag{8}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\,e^{-ikx}e^{-ax^2}. \tag{9}$$

5) Total Differentials

a) Calculate the total differential of the function

$$F(x, y, z,) = x^4 y^3 + zx + z^2 y. \tag{10}$$

b) Now, for which numerical value of $a \in \mathbb{R}$ is the following expression a total differential,

$$\mathrm{d}G = axyz\,\mathrm{d}x + x^2 z\,\mathrm{d}y + x^2 y\,\mathrm{d}z, \tag{11}$$

i.e. for which a does a function $G(x, y, z)$ exist?

6) Lagrange Multipliers

Find the maxima of the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 - y^2$ along the unit circle, i.e. along $x^2 + y^2 = 1$. In general, how do additional constraints enter the Lagrange function?