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Advanced Statistical Physics II – Supplement for Problem Sheet 7

1 Line integrals

Recall the definition of a line integral in the complex plane along a curve
$\gamma : [a, b] \ni s \mapsto \gamma(s) \in \mathbb{C}$

$$\int_\gamma dz f(z) := \int_a^b ds f(\gamma(s)) \frac{d\gamma(s)}{ds} \tag{1}$$

The curve is called *simple* if it does not intersect itself (The curves we will be considering are all simple). If the curved is *closed*, its start and end point coincice, i.e. $\gamma(a) = \gamma(b)$, and we use the following notation for integrals along this curve:

$$\oint_\gamma dz f(z) \tag{2}$$

As an example, consider a (closed) curve describing a circle of radius r around the origin.

$$\gamma_+^r(s) = re^{is}, \qquad s \in [0, 2\pi] \tag{3}$$

Note that the circle is traversed counter-clockwise ("positive orientation"), while a different parametrization

$$\gamma_-^r(s) = re^{-is}, \qquad s \in [0, 2\pi] \tag{4}$$

gives a negatively oriented circle. The orientation of curve is important for getting the correct signs of expressions when doing contour integration. Consider the function $f(z) = 1/z$.

$$\oint_{\gamma_+^r} dz f(z) = \oint_{\gamma_+^r} \frac{dz}{z} = \int_0^{2\pi} ds \frac{ire^{is}}{re^{is}} = 2\pi i \tag{5}$$

while for the negative (clockwise) orientation we get

$$\oint_{\gamma_-^r} dz f(z) = \oint_{\gamma_-^r} \frac{dz}{z} = \int_0^{2\pi} ds \frac{-ire^{-is}}{re^{-is}} = -2\pi i \tag{6}$$

For the case $f(z) = z^n$ where $n \in \mathbb{Z}$ is now an arbitrary integer, we get by a similar calculation

$$\oint_{\gamma_\pm^r} dz f(z) = \oint_{\gamma_\pm^r} \frac{dz}{(z - z_0)^n} = \begin{cases} \pm 2\pi i, & n = 1 \\ 0, & \text{else} \end{cases} \tag{7}$$

where the circle $\gamma_\pm^r$ is now centered at z_0, i.e $\gamma_\pm^r(s) = z_0 + re^{\pm is}$.
Now consider a function of the form

$$f(z) = \frac{g(z)}{(z - z_0)^n} \tag{8}$$

where g is a holomorphic function, which means that it can be expanded in a power series around every point z_0 in the complex plane:

$$g(z) = a_{-n} + \dots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \dots \tag{9}$$

$$\Rightarrow \quad f(z) = \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots \tag{10}$$

Integrating f around a circle centered at z_0 we get by application of (7)

$$\oint_{\gamma^r_\pm} dz f(z) = \pm 2\pi i a_{-1} \tag{11}$$

On the other hand

$$a_{-1} = \frac{1}{(n-1)!} \left. \frac{d^{n-1} g(z)}{dz^{n-1}} \right|_{z=z_0} = \frac{1}{(n-1)!} \left. \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \right|_{z=z_0} = \mathrm{Res}(f, z_0) \tag{12}$$

where we introduced the *residue* $\mathrm{Res}(f, z_0)$, which is just another name for the expansion coefficient a_{-1}. For the case of a positively oriented circle we get

$$\frac{1}{2\pi i} \oint_{\gamma^r_+} dz f(z) = \mathrm{Res}(f, z_0) \tag{13}$$

Remark Everything we did above for circles, also holds for arbitrary simple closed curves, as long as they are sufficiently smooth etc. (See a textbook on complex analysis for the mathematical details). Eventually, this leads to the Residue theorem:

$$\oint_\gamma dz f(z) = 2\pi i \sum_{k=1}^n \mathrm{Res}(f, z_k) \tag{14}$$

where the simple, closed curve γ winds counter-clockwise around a region containing the poles z_k. In the more general case, in which γ may be also negatively oriented we get

$$\oint_\gamma dz f(z) = \pm 2\pi i \sum_{k=1}^n \mathrm{Res}(f, z_k) \tag{15}$$

where we have $+$ for positive and $-$ for negative orientation of γ.

Watch out for the orientation of the curves when doing the problems!