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Second Chance Exam Solutions: Advanced Statistical Physics
Part II: Problems (75P)

1 Three-Spin Interaction (20P)

Consider a one dimensional system of N spins with the following Hamiltonian:

$$H = -J \sum_{i=1}^{N-2} S_i \, S_{i+1} \, S_{i+2},$$

with $S_i = \pm 1, i = 1...N$ being the spin states and $J = const$ being an interaction parameter. Assume open boundary conditions (**not** periodic!).

Calculate the canonical partition function and the Helmholtz free energy F. What is the thermodynamic limit of F?

*Hint: The transfer-matrix method is **not** necessary for this problem.*

The definition of the cosinus hyperbolicus may be helpful:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\begin{aligned}
Z_N &= \sum_{S_1,\ldots,S_N} e^{\beta\, J \sum_{i=1}^{N-2} S_i\, S_{i+1}\, S_{i+2}} \\
&= \sum_{S_1,\ldots,S_{N-1}} e^{\beta\, J \sum_{i=1}^{N-3} S_i\, S_{i+1}\, S_{i+2}} \sum_{S_N} e^{\beta\, J\, S_{N-2}\, S_{N-1}\, S_N} \\
&= \sum_{S_1,\ldots,S_{N-1}} e^{\beta\, J \sum_{i=1}^{N-3} S_i\, S_{i+1}\, S_{i+2}}\, 2\cosh(\beta\, J) \\
&= \sum_{S_1,S_2,S_3} e^{\beta\, J\, S_1\, S_2\, S_3} \Big(2\cosh(\beta\, J)\Big)^{N-3} \\
&= 4\Big(2\cosh(\beta\, J)\Big)^{N-2}
\end{aligned}$$

Therefore the Helmholtz free energy is:

$$F = -\frac{1}{\beta}\ln Z_N = -\frac{1}{\beta}\Big(\ln 4 + (N-2)\ln(2\cosh(\beta\, J))\Big)$$

For the thermodynamic limit we get:

$$\lim_{N\to\infty} F = -\frac{N}{\beta}\ln(2\cosh(\beta\, J))$$

2 Liquid-Gas Phase-Transition (25P)

We want to consider a substance with the enthalpy of the liquid phase

$$H_l(p,S) = 2\,(a\,p\,S\,N)^{1/2}$$

and the enthalpy of the gas phase

$$H_g(p,S) = 3\left(\tfrac{p\,S}{2}\right)^{2/3}(b\,N)^{1/3},$$

where $a > 0$ and $b > 0$ are constants, S is the total entropy of the system, p is the pressure and N is the total number of particles.

a) Calculate the liquid-gas coexistence temperature T_g as a function of pressure. (16P)

We have to perform a Legrende-Transformation from $H(S,p)$ to $G(T,p) = H - TS$ with $T = \partial H/\partial S$:

liquid phase:

$$\begin{aligned}
T &= \partial H_l/\partial S = \sqrt{\frac{aNp}{S}} && (1)\\
\Rightarrow S &= \frac{aNp}{T^2} && (2)\\
G_l(T,p) &= 2aN\frac{p}{T} - aN\frac{p}{T} = aN\frac{p}{T} && (3)\\
g_l(T,p) &= G_l/N = a\frac{p}{T} && (4)
\end{aligned}$$

gas phase:

$$\begin{aligned}
T &= \partial H_g/\partial S = 2\left(\frac{p}{2}\right)^{2/3}\left(\frac{bN}{S}\right)^{1/3} && (5)\\
\Rightarrow S &= 2bN\frac{p^2}{T^3} && (6)\\
G_g(T,p) &= 3bN\frac{p^2}{T^2} - 2bN\frac{p^2}{T^2} = bN\frac{p^2}{T^2} && (7)\\
g_g(T,p) &= G_g/N = b\frac{p^2}{T^2} && (8)
\end{aligned}$$

At the phase coexistence line we have:

$$\begin{aligned}
g_l(T_v,p_v) &= g_g(T_v,p_v) && (9)\\
\Rightarrow a\frac{p_v}{T_v} &= b\frac{p_v^2}{T_v^2} && (10)\\
\Rightarrow T_v &= p_v\frac{b}{a} && (11)
\end{aligned}$$

b) Calculate the densities of the liquid and the gas phase at the phase transition line. (6P)

liquid phase:

$$\begin{aligned} V_l &= \partial G_l/\partial p = \frac{aN}{T_v} && (12) \\ \Rightarrow \rho_l &= N/V_l = \frac{T_v}{a} && (13) \end{aligned}$$

gas phase:

$$\begin{aligned} V_g &= \partial G_g/\partial p = \frac{2bNp_v}{T_v^2} \overset{(11)}{=} \frac{2aN}{T} && (14) \\ \Rightarrow \rho_g &= N/V_g = \frac{T_v}{2a} && (15) \end{aligned}$$

c) Calculate the entropy change per volume $\Delta S/\Delta V$ at the phase transition line. (3P)

We can use Clausius-Clapeyron:

$$\frac{\Delta S}{\Delta V} = \frac{\Delta S/N}{\Delta V/N} = \Delta s/\Delta v = \frac{\mathrm{d}p_v}{\mathrm{d}T_v} \overset{(11)}{=} \frac{a}{b} \tag{16}$$

3a Wien's Law in n Dimensions (15P)

$u(\omega)$ is the energy density of black body radiation at angular frequency ω per volume and angular frequency, i.e. the spectral density of the internal energy density U/V.

Determine $u(\omega)$ in n dimensions in the low temperature limit.

The volume of an n-dimensional sphere is $C_n R^n$, where R is the radius. You do not have to determine C_n.

$$U = 2\frac{V}{h^n}\int \mathrm{d}^n p \, \frac{e^{-\beta pc}}{1-e^{-\beta pc}} pc \tag{17}$$

$$= 2\frac{V}{h^n} C_n \int \mathrm{d}p \, \frac{e^{-\beta pc}}{1-e^{-\beta pc}} n p^{n-1} pc \tag{18}$$

$$= 2n\frac{cV}{h^n} C_n \int \mathrm{d}p \, \frac{e^{-\beta pc}}{1-e^{-\beta pc}} p^n \tag{19}$$

$$pc = \hbar\omega \Rightarrow \mathrm{d}p = \frac{\hbar}{c}\mathrm{d}\omega \tag{20}$$

$$\Rightarrow U = 2n\frac{cV}{h^n} C_n \int \mathrm{d}\omega \, \frac{\hbar}{c} \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \left(\frac{\hbar\omega}{c}\right)^n \tag{21}$$

$$= 2n\frac{V\hbar}{(2\pi c)^n} C_n \int \mathrm{d}\omega \, \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \omega^n \tag{22}$$

For $\beta\hbar\omega \gg 1$ we can approximate $1 - e^{-\beta\hbar\omega} \approx 1$:

$$\Rightarrow U = 2n\frac{V\hbar}{(2\pi c)^n} C_n \int \mathrm{d}\omega \, e^{-\beta\hbar\omega} \omega^n \tag{23}$$

$$\frac{U}{V} = \int \mathrm{d}\omega \, u(\omega) \tag{24}$$

$$\Rightarrow u(\omega) = 2n\frac{\hbar}{(2\pi c)^n} C_n e^{-\beta\hbar\omega} \omega^n \tag{25}$$

3b Cosmic Background Radiation (15P)

The universe is filled with a photon gas that corresponds to black body ratiation of temperature $T_{present} = 3\,\mathrm{K}$. In a simple view, this radiation arose from the isentropic expansion of a much hotter photon cloud, which was produced during the big bang.

If the volume of the universe, and thus the volume of the photon gas, increases isentropically by a factor of two starting from the present state, what will be the final temperature of the photon gas?

Hint: The Stefan-Boltzmann law can be useful.

Stefan-Boltzmann law: $U \propto VT^4$

The total differential is

$$\begin{aligned}
\mathrm{d}U &= T\mathrm{d}S + p\mathrm{d}V && (26)\\
\Rightarrow T &= \left(\frac{\partial U}{\partial S}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial S}\right)_V && (27)\\
\Rightarrow \left(\frac{\partial S}{\partial T}\right)_V &= \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V && (28)\\
\Rightarrow S \propto VT^3 & && (29)
\end{aligned}$$

isentropic: dS = 0

$$\begin{aligned}
\Rightarrow V_1 T_1^3 &= V_2 T_2^3 && (30)\\
\Rightarrow T_2 &= T_1 \left(V_1/V_2\right)^{1/3} = T_1 2^{-1/3} && (31)
\end{aligned}$$