Freie Universität Berlin
Fachbereich Physik
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Statistical Physics and Thermodynamics (SS 2017)

Problem sheet 1

Hand in: Friday, April 28 during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1 Yahtzee (5 points)

We want to consider a simplified version of the game of dice Yahtzee: 5 ideal dice are thrown at the same time only once per move. Calculate:

a) The probability of a Yahtzee, which is all five dice showing the same face. **(1 point)**

b) The probability of a Large Straight, which is five sequential dice (1-2-3-4-5 or 2-3-4-5-6). **(1 point)**

c) The probability of a Full House, i. e. a three-of-a-kind and a pair with a different face. **(2 points)**

d) The probability of exactly one (and only one) Yahtzee in ten moves. **(1 point)**

2 Binomial distribution (8 points)

A coin is thrown N times. The probability for heads is $p \in [0, 1]$ and the probability for tails is $1-p$. If $p = 1/2$, we call the coin fair, if $p \neq 1/2$, we call the coin unfair.

a) What is the probability $P_N(k)$ for obtaining heads exactly k times? **(2 points)**

b) Show that the probability distribution from task a) is properly normalized, i.e. that

$$\sum_{k=0}^{N} P_N(k) = 1. \tag{1}$$

(2 points)

Hint: Use the binomial formula.

c) Now assume every time the coin shows heads, you gain 1 Euro, and every time the coin shows tails, you lose 1 Euro. Let X be your balance after N coin tosses. Calculate the expectation value

$$\langle X \rangle = \sum_{k=0}^{N} x(k) P_N(k), \tag{2}$$

where $x(k)$ is how much money you win if heads shows up exactly k times. Are your results for a fair coin ($p = 1/2$) and a maximally unfair coin ($p = 0$ and $p = 1$) what you would intuitively expect (and if yes or no, why)? **(3 points)**

Hint: Use the identity

$$n\, p^{n-1} = \frac{\partial}{\partial p} p^n. \tag{3}$$

d) Now consider you throw a die (Würfel) 10 times and every time one of the numbers 3, 4, 5 or 6 show up, you win 1 Euro, while if the numbers 1 or 2 come up, you lose 1 Euro. What is the expectation value of your net money earning? **(1 point)**

Hint: Assume the die to be fair, i.e. that each number shows up with probability $1/6$.

3 Poisson distribution (7 points)

The Poisson distribution describes the probability for the number of independent rare events, and in this exercise we will derive it as an approximation to the binomial distribution.

Assume you work in a call center, and you get on average α calls per unit time, so that the probability to get a call in a time interval Δt is $p_{\Delta t} = \alpha \cdot \Delta t$. We assume that there are so few callers that we neglect the possibility of two or more callers per time interval Δt. The total duration of your working shift is $T = N \cdot \Delta t$, with N the number of intervals Δt during your working shift.

The probability to get exactly k calls during your shift is then given by the binomial distribution

$$P_N(k) = \binom{N}{k} p_{\Delta t}^k (1 - p_{\Delta t})^{N-k}. \tag{4}$$

a) Show that in the limit $p_{\Delta t} \to 0$, $N \to \infty$ such that $p_{\Delta t} \cdot N = \alpha \cdot T$ remains constant, the binomial distribution can be approximated by the Poisson distribution, i.e.

$$\lim_{p_{\Delta t} \to 0} P_N(k) = \frac{\lambda^k}{k!} \exp(-\lambda) \tag{5}$$

with $\lambda = \alpha T$. **(3 points)**

Hint: Use that $\lim_{N\to\infty}(1 - \lambda/N)^N = \exp(-\lambda)$.

b) Show that the probability distribution (5) is properly normalized, i.e. that

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \exp(-\lambda) = 1. \tag{6}$$

(1 point)

c) Show that the expectation value of the total number of calls is given by $\langle k \rangle = \alpha T$. **(1 point)**

Now assume you get on average 7 calls during your whole shift.

d) What is the probability to get 0 calls during your whole shift? **(1 point)**

e) What is the probability to get more than two calls calls during your whole shift? **(1 point)**