Freie Universität Berlin
Fachbereich Physik
April 18th 2017

Prof. Dr. Roland Netz
Douwe Bonthuis
Julian Kappler
Philip Loche

Statistical Physics and Thermodynamics (SS 2017)

Problem sheet 2

Hand in: Friday, May 5 during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1 Characteristic Functions (10 points)

Consider the normalized probability distributions

$$P_1(x) = A_1 \delta(x - x_0), \tag{1}$$

$$P_2(x) = A_2 \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{else,} \end{cases} \tag{2}$$

$$P_3(x) = A_3 \mathrm{e}^{-a|x|}, \tag{3}$$

$$P_4(x) = A_4 \frac{1}{1 + x^2}, \tag{4}$$

where $\delta(x)$ is the Dirac-delta distribution, $x, x_0, a \in \mathbb{R}$ and $a > 0$.

a) Determine the normalization constants $A_1, \dots A_4$. **(1 point)**

b) Calculate the characteristic function (moment generating function) $G(k) = \langle e^{-ikx} \rangle = \int_{-\infty}^{\infty} P(x) \mathrm{e}^{-ikx} \mathrm{d}x$ and the cumulant generating function $\ln G(k)$ for all distributions. **(6 points)**

Hint: For calculating $G_4(k)$ you can either use the "Residue theorem" or the relation of $P_3(x)$ and $G_3(k)$.

c) Calculate the moments $\langle x^n \rangle$ and the cumulants $\langle x^n \rangle_c$ for $n = 1, 2$ from the generating functions for all the distributions from part a). What are the relations among moments, cumulants, mean and variance? **(3 points)**

2 Central Limit Theorem (10 points)

In the following we investigate the central limit theorem in more detail. The central limit theorem states that the distribution of an average of random variables approaches a Gaussian, even if the individual random variables are not distributed according to a Gaussian.

a) Consider N equally distributed random variables $x_i,\ i \in \{1, \dots, N\}$, and their average $X = 1/N \sum_{i=1}^{N} x_i$. In the lecture we derived the relation between the cumulants of X and x

$$\langle X^m \rangle_c = \frac{1}{N^{m-1}} \langle x^m \rangle_c. \tag{5}$$

Expand $\ln G(k)$ in a power series in k and use equation (5) to express this power series in terms of the cumulants of x. **(2 points)**

b) Take the limit $N \to \infty$ of $\ln G(k)$ and use inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \mathrm{d}k, \tag{6}$$

to show that $P(X)$ is the Dirac-delta distribution. **(2 points)**

Remark: This is the law of large numbers. If we average over infinitely many samples, we will for sure get the expectation value.

c) Now we want to see how the distribution narrows as we average over larger and larger sample sizes. Only keep the leading order term in $1/N$ in your expansion from a) and show that the resulting $P(X)$ is Gaussian. How are standard deviation and mean of x_i related to standard deviation and mean of the Gaussian distribution? What happens to the Gaussian distribution in the limit $N \to \infty$? **(3 points)**

Remark: This is the central limit theorem. As the sample size is increased, the higher cumulants of the average become less and less relevant, so that the distribution approaches a Gaussian.

d) Imagine you observe the positions of ants, and that these positions are uniformly distributed within a distance ± 5 m around their nest entry. How many ants do you have to observe to know the position of the nest entry up to a standard deviation of 1 mm? **(1 point)**

Hint: Assume the ants positions are all independent and that the ants live in a 1 dimensional world.

e) Now consider the Lorentz distribution

$$P(x_i) = \frac{1}{\pi} \frac{1}{1 + x_i^2} \tag{7}$$

and use the characteristic function from exercise 1 to calculate $G(k) = \langle e^{-ikX} \rangle$ for the average over N independent random variables $X = 1/N \sum_{i=1}^{N} x_i$.

Does the distribution of X approach a Gaussian as N goes to infinity? Are your observations in contradiction to the central limit theorem? **(2 points)**

Hint: What are the requirements on $G(k)$ in the derivations you carried out in parts a) and b)?