Freie Universität Berlin
Fachbereich Physik
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Statistical Physics and Thermodynamics (SS 2017)

Problem sheet 3

Hand in: Friday, May 12 during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1. Central moments of a discrete probability distribution (3 points)

Consider a discrete probability distribution $P(m)$, with $m \in \mathbb{Z}$. For any observable $A(m)$ the relation $\langle A(m) \rangle = \sum_m A(m)P(m)$ holds.

a) Show that $\langle (m - \langle m \rangle)^2 \rangle = \langle m^2 \rangle - \langle m \rangle^2$. **(1 point)**

b) Calculate the central moments $\langle (m - \langle m \rangle)^n \rangle$ for $n = 3$ and $n = 4$ in terms of the moments of m. **(2 points)**

2. Moments and cumulants of the Poisson distribution (10 points)

Consider the number of messages m sent between two nodes on a network within a time between 0 and t. The number m can take any non-negative integer value. Assume that the values of m are distributed according to a Poisson distribution,

$$P_{\alpha t}(m) = \frac{(\alpha t)^m \, e^{-\alpha t}}{m!}, \tag{1}$$

with α being the average number of messages sent per unit time.

a) Calculate the characteristic function $G(k)$ of the Poisson distribution. **(2 points)**
Hint: Use the relation $e^y = \sum_{i=0}^{\infty} y^i / i!$.

b) Calculate the moments $\langle m^n \rangle$ for $n = 1, 2, 3, 4$ from $G(k)$. **(2 points)**

c) Using your results from exercise 1, calculate the central moments $\langle (m - \langle m \rangle)^n \rangle$ for $n = 1, 2, 3, 4$. **(4 points)**

d) From the cumulant generating function, $\ln G(k)$, calculate the cumulants $\langle m^n \rangle_C$ for $n = 1, 2, 3, 4$. At which value of n does your answer differ from the answer to part (c)? **(2 points)**

3. Joint probability density function (7 points)

Now the observation time t considered in exercise 2 is split into two parts, $t = t_1 + t_2$.

a) Insert $t = t_1 + t_2$ into the distribution of Eq. 1 and rewrite $P_{\alpha t}(m)$ in the form of a sum over the joint probability function $W(m_1, m_2)$:

$$P_{\alpha t}(m) = \sum_{m_1=0}^{m} W(m_1, m_2), \tag{2}$$

with m_1 being the number of messages sent in the time t_1, m_2 being the number of messages sent in the time t_2, and $m = m_1 + m_2$. Give the expression for $W(m_1, m_2)$ as a function of t_1 and t_2. **(4 points)**

b) Rewrite the resulting $W(m_1, m_2)$ from part (a) in terms of two Poisson distributions with the appropriate arguments and parameters. **(1 point)**

c) Calculate the covariance

$$\langle m_1 m_2 \rangle - \langle m_1 \rangle \langle m_2 \rangle.$$

Note that your result means that m_1 and m_2 are uncorrelated variables. **(2 points)**

Hint: Write down the characteristic function $G(k_1, k_2)$ of the right-hand side of Eq. 2 in terms of t_1 and t_2, noting that the expresion in Eq. 2 constitutes the convolution of the two Poisson distributions derived in part (b).