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# Advanced Statistical Physics II – Problem Sheet 4

Problem 1 – **Green's functions**
The Green's function $G(\vec{x})$ of a linear differential operator $\mathcal{L}$ with constant coefficients is defined via

$$\mathcal{L}G(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y}), \qquad \vec{x},\vec{y} \in \mathbb{R}^n \tag{1}$$

For arbitrary inhomogeneity $f(\vec{x})$, the solution of $\mathcal{L}\rho(\vec{x}) = f(\vec{x})$ is given by

$$\rho(\vec{x}) = \int d^n x' G(\vec{x}-\vec{x}')f(\vec{x}') \tag{2}$$

since

$$\mathcal{L}\rho(\vec{x}) = \int d^n x' f(\vec{x}')\mathcal{L}G(\vec{x}-\vec{x}') = \int d^n x' f(\vec{x}')\delta(\vec{x}-\vec{x}') = f(\vec{x})$$

a) (4P) Consider the (inhomogeneous) diffusion equation in one dimension:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)\rho(x,t) = f(x,t) \tag{3}$$

Using Fourier analysis, show that the Green's function $G(x,t)$ for eq. (3) is given by

$$G(x,t) = \frac{\Theta(t)}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}, \qquad \Theta(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{else} \end{cases} \tag{4}$$

Remember problem sheet number 0!
b) (4P) Use $G(x,t)$ to find the solution $\rho(x,t)$ for a box-shaped initial density profile:

$$\rho_0(x) = \begin{cases} 1/2l & |x| < l \\ 0 & \text{else} \end{cases} \tag{5}$$

i.e. $f(x,t) = \delta(t)\rho_0(x)$. Express the solution in terms of the error function

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \tag{6}$$

*Suggestion:* It is instructive to plot the solution $\rho(x,t)$ vs. the position $x$ for various times $t$. (This is not mandatory.)

Problem 2 – **Time-dependent observables**

a) (1P) Consider a probability density on phase space which only depends on the sum of the system's Hamiltonian and a perturbation, i.e. $\rho \equiv \rho(\mathcal{H}(q,p) + \Delta(q,p))$. Show that such a distribution is not stationary for arbitrary $\Delta(q,p)$.

b) (1P) In the lecture, we derived the formula for the time correlation function of two observables $A(q,p)$ and $B(q,p)$ for a stationary system:

$$C_{AB}(\tau) = \int \mathrm{d}q\mathrm{d}p\mathrm{d}q'\mathrm{d}p' A(q,p)B(q',p')\mathrm{e}^{-\tau L(q,p)}\delta(q-q')\delta(p-p')\rho_0(q',p') \tag{7}$$

Here, $L$ is the Liouville operator, $\rho_0 \equiv \rho_0(\mathcal{H}(q',p'))$ is a stationary density distribution and we assume that $\mathcal{H}(q,-p) = \mathcal{H}(q,p)$ (since we assume that $\mathcal{H}(q,p) \sim p^2$). Recall from the lecture that the substitution $p \to -p, \quad p' \to -p'$ corresponds to time reversal. Find a relation between $C_{AB}(\tau)$ and $C_{AB}(-\tau)$ for the following observables:

i) $A = q$ and $B = p^2$

ii) $A = p$ and $B = H(q,p)$

iii) $A = p$ and $B = qp^3$

Problem 3 – **Some probability theory**

a) (2P) Assume that 80% of all emails are spam. 10% of all spam-mails contain the phrase "business proposal" while only 1% of the non-spam ones do. What is the probability that a given message containing the phrase "business proposal" is spam?

b) (4P) Consider the bivariate Gaussian distribution

$$p(x,y) = \frac{\exp\left(-\frac{1}{2}(\vec{u}-\vec{\mu})^T\Sigma^{-1}(\vec{u}-\vec{\mu})\right)}{\sqrt{4\pi^2|\Sigma|}}, \qquad \vec{u}^T = (x,y) \tag{8}$$

with mean and covariance matrix:

$$\vec{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix}. \tag{9}$$

Use the Chapman-Kolmogorov equation to calculate the marginal distributions $p_x(x)$ and $p_y(y)$.

c) (4P) Calculate the conditional distribution $p(x|y)$ for (8) (which is again Gaussian). Compute its mean and variance.