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Advanced Statistical Physics II – Problem Sheet 7

Problem 1 – **Residual calculus**
a) (2P) Show that

$$\oint_\gamma \frac{dz}{(z-z_0)^k} = \begin{cases} 2\pi i, & k=1 \\ 0, & \text{else} \end{cases} \tag{1}$$

where $k \in \mathbb{N}$, and the simple, closed, positively oriented (i.e. counter-clockwise direction of traversal) curve γ around z_0.
b) (3P) Now consider a function f which has a pole of order k at z_0, i.e. in some small enough region around z_0, we may find a function g, such that $f(z) = \frac{g(z)}{(z-z_0)^k}$ where g can be expanded in a power series. The *residue* of f at z_0 is defined as

$$\mathrm{Res}(f, z_0) := \frac{1}{2\pi i} \oint_\gamma dz f(z) \tag{2}$$

where γ is a sufficiently "small" simple, closed, positively oriented curve around z_0. Show that

$$\mathrm{Res}(f, z_0) = \frac{1}{(k-1)!} \frac{d^{k-1}}{z^{k-1}} (z-z_0)^k f(z) \bigg|_{z=z_0} \tag{3}$$

by writing $f(z)$ as $f(z) = \frac{g(z)}{(z-z_0)^k}$, expanding $g(z)$ as a power series around z_0 and using (1).
c) (2P) Convince yourself, that using the convention $\tilde{\chi}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \chi(t)$, the Fourier transform of $\chi(t) = \Theta(t) e^{-t/\tau}, \quad (\tau > 0)$ is given by

$$\tilde{\chi}(\omega) = \frac{1}{\tau^{-1} - i\omega} \tag{4}$$

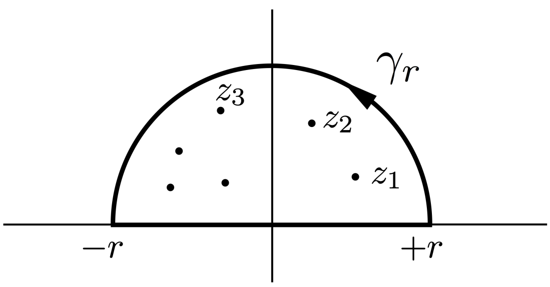
How does the pole-structure of $\tilde{\chi}(\omega)$ reflect the fact that $\chi(t)$ is a single-sided function?

Problem 2 – **Contour integration**
Recall the residue theorem, which states that

$$\oint_\gamma dz f(z) = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

where the simple, closed curve γ winds counter-clockwise around a region containing the poles z_k. This may be used to evaluate integrals $\int_{-\infty}^{\infty} dx g(x)$ by considering the contour in the figure to the right.

$$2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k) = \oint_\gamma dz f(z) = \int_{-r}^{r} dx f(x) + \int_{\gamma_r} dz f(z) \tag{5}$$



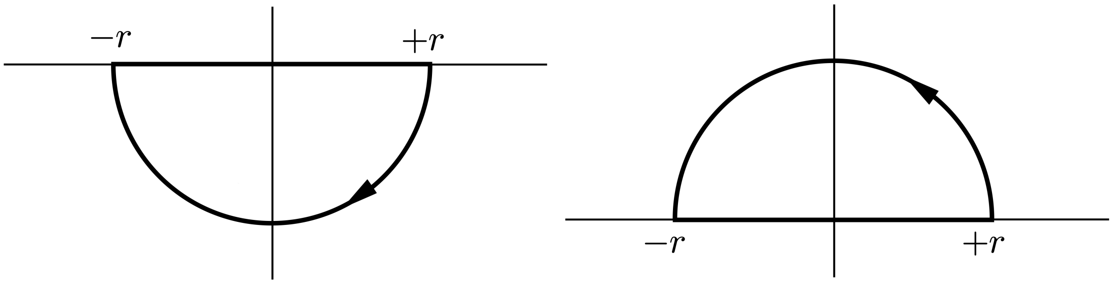


Figure 1: Left: A contour closed in the lower half plane. Right: A contour closed in the upper half plane

If the contribution of "the arc" γ_r vanishes in the limit $r \to \infty$ and g has no poles on the real line, then $\int_{-\infty}^{\infty} dx f(x) = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$.
a) (3P) Consider a function $f(z)$ on the complex plane wich decays faster than $|z|^{-1}$ for large $|z|$, i.e $\lim_{|z|\to\infty} f(|z|e^{i\varphi})|z| = 0$. Consider an integral along an arc γ_r of radius r in the upper half plane, i.e. $\gamma_r(\varphi) = re^{i\varphi}$ with $0 \le \varphi \le \pi$.

$$\int_{\gamma_r} dz f(z) = \int_0^{\pi} d\varphi f(\gamma_r(\varphi))\dot{\gamma}_r(\varphi) \tag{6}$$

Show that the integral goes to zero in the limit of $r \to \infty$ by considering $\left|\int_{\gamma_r} dz f(z)\right|$ and finding an upper bound which goes to zero. You may assume, that f takes on a maximum $m(r)$ on an arc of radius r.

b) (5P) Do the back-transform of the single sided exponential $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\tau^{-1}-i\omega}$ via contour integration: Consider the cases $t > 0$ and $t < 0$ separately: For each case, close the contour either in the upper or lower half plane (see fig. 1) such that the contribution from the arc gives zero. Explicitly show that the arc-contribution indeed vanishes by using the line of argument of the previous example.

Problem 3 (5P) – **Principal value integrals - poles on the real line**
Consider the function $\tilde{\chi}(\omega) = \frac{1}{\tau^{-1}-i\omega}$ with real and imaginary parts $\tilde{\chi}'(\omega)$ and $\tilde{\chi}''(\omega)$, respectively. Verify the Kramers-Kronig relation

$$\tilde{\chi}'(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}''(\omega')}{\omega' - \omega} \tag{7}$$

by computing the principal value integral in (7) by "lifting" the pole at $\omega' = \omega$ via the following regularization scheme:

$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}''(\omega')}{\omega' - \omega} = \lim_{\epsilon\to 0}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}''(\omega')}{\omega' - \omega - i\epsilon} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}''(\omega')}{\omega' - \omega + i\epsilon}\right] \tag{8}$$

Before taking the limit $\epsilon \to 0$, the integrand has no poles on the real line. Close each contour in (8) in either the lower or upper half plane, such that it encloses the lifted pole $\omega \pm i\epsilon$.