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Advanced Statistical Physics II – Problem Sheet 11

Problem 1 – Path Integral formulation of Langevin equation
Consider the stochastic path integral derived during the lecture

$$\tilde{P}_T(x_T) = \int \mathcal{D}x(\cdot)e^{-D\mathcal{S}[x(\cdot)]}\tilde{P}_0(x_0) \tag{1}$$

where $\mathcal{S}$ is the Lagrangian action,

$$\mathcal{S} = \int_0^T dt \left[\frac{\dot{x}^2(t)}{4D^2} + U_{eff}\big(x(t)\big)\right] \tag{2}$$

and $U_{eff}(x)$ is the effective potential

a) (5P) Derive the Newton equation

$$U'_{eff}\big(x(t)\big) - \frac{\ddot{x}(t)}{2D^2} = 0 \tag{3}$$

by minimizing the action.

Hint: $\frac{\delta\mathcal{S}[x(t)]}{\delta x(\tilde{t})} = \left.\frac{\mathcal{S}[x(t)+\epsilon\delta(t-\tilde{t})]-\mathcal{S}[x(t)]}{\epsilon}\right|_{\epsilon\to 0}$.

b) (3P) Now consider a Langevin equation with potential $U(x) = -\frac{k}{2}x^2, k > 0$ and use the expression for the effective potential

$$U_{eff}(x) = \left[\frac{U'(x)}{2k_BT}\right]^2 - \frac{U''(x)}{2k_BT}. \tag{4}$$

Solve eq. 3 by using the Ansatz

$$x(t) = Ae^{\frac{k}{\gamma}t} + Be^{-\frac{k}{\gamma}t}, \tag{5}$$

the effective potential in eq. (4) and the boundary conditions $x(0) = -x_0$ and $x(T) = x_0$. Find A and B. Remember the Einstein relation $D = k_BT/\gamma$.

c) (3P) Sketch $x(t)$ in the limit $\frac{kT}{\gamma} \gg 1$.

d) (4P) By inserting your solution into the action $\mathcal{S}$, derive the following expression for $\mathcal{S}$

$$\mathcal{S} = \frac{kx_0^2}{2\gamma D^2}\left[\frac{DT}{x_0^2} + \frac{e^{\frac{k}{\gamma}T}+1}{e^{\frac{k}{\gamma}T}-1}\right] \tag{6}$$

e) (5P) Find the optimal transition path time T^* by minimizing the action in eq. 6 in the high barrier limit $kT/\gamma \gg 1$ and compare with the Kramers time.