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Fachbereich Physik
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Advanced Statistical Physics (WS11/12)
Problem sheet 12

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Problem 1: Scaling Hypothesis

The scaling hypothesis assumes that the free energy per volume of a magnetic system is a homogeneous function of the reduced temperature $t = \frac{T-T_c}{T_c}$, a length rescaling factor b and the reduced magnetic field h:

$$\frac{F}{k_B TV} = \frac{f}{k_B T} = b^{-d} g(b^{1/\nu} t, b^c h),$$

where d is the system dimensionality, ν is the critical exponent of the correlation length, $\xi \propto t^{-\nu}$, and c an unknown exponent.

a) Calculate the critical exponent β of the spontaneous magnetisation $m = \partial f/\partial h \propto t^\beta$ for $h = 0$.

b) Calculate the critical exponent γ of the susceptibility $\chi = \partial m/\partial h \propto t^{-\gamma}$ for vanishing external field $h = 0$.

c) Use the results of a) and b) to eliminate the constant c and derive a relation between the exponents β and γ.

Problem 2: Vibrational Modes

We want derive an expression for the vibrational specific heat C_{vib} of a single diatomic gas molecule as a function of temperature T. The vibration energy levels are

$\epsilon_k = \hbar\omega_0(k + 1/2)$, for $k = 0, 1, 2, ...$

a) Calculate the partition function.

b) Calculate the free energy.

c) Calculate the internal energy.

d) Finally derive the vibrational specific heat C_{vib} and discuss the limit of high temperature as well as the limit of low temperature.

Hint: You can use $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ for $0 < q < 1$.

Problem 3: Two Particle Wave Function

We want to discuss a system of two particles of mass m in a one-dimensional box of length L. The probability density $|\psi^2|$ vanishes for the particles at the box walls.

a) Use the single particle wave functions $u_n(x)$ to construct the explicit two-particle wave functions $\psi_{\text{n,m}}(x, y)$ for bosonic particles and (spinless) fermions.

b) Calculate the ground state energy and the energy of the first two excited states for bosonic particles and (spinless) fermions.

c) Show that the expectation value of the inter particle distance $\langle x-y \rangle$ vanishes. Calculate the second momentum $\langle (x - y)^2 \rangle$ of the particle distance for bosonic particles and (spinless) fermions in the ground state. Compare your result with the one you would obtain for classical particles and give a short interpretation of the differences.

Problem 4: Kramers-Wannier-Duality in the 2D-Ising Model

We want to calculate the critical temperature T_c of the two-dimensional Ising model defined by the Hamilitonian $H = -J \sum_{\text{i,j}} s_i s_j$, where the sum includes the nearest horizontal and vertical neighbours with $s_i = \pm 1$.

a) Explain in words that the partition function for low temperature ($K^* = J/(k_B T^*) \gg 1$) can be writen as $Z(K^*) = 2e^{2K^*N}\left[1 + \sum_{n_\mathcal{D}} \nu(n_\mathcal{D}) e^{-2K^* n_\mathcal{D}}\right]$, where $n_\mathcal{D}$ is the sum of all perimeters of a configuration of polygons and $\nu(n_\mathcal{D})$ the number of configurations of polygons a with perimeter sum $n_\mathcal{D}$ (see Fig. 1).
(The partition function $Z(K^*)$ is not only an approximation for low temperature. In fact the above expression for $Z(K^*)$ is exact for all temperatures.)

b) Show that the following approximation for the partition function is valid for high temperature ($K = J/k_B T \ll 1$):

$$Z(K) = 2^N (\cosh K)^{2N} \left[1 + \sum_{n_\mathcal{D}} \nu(n_\mathcal{D}) (\tanh K)^{n_\mathcal{D}}\right]$$

Hint: You can use $e^{s_i s_j K} = \cosh(K) + s_i s_j \sinh(K)$, since $s_i s_j = \pm 1$. Show which class of terms in the partition function will cancel each other and which will survive.

c) The 2D-Ising model has a symmetry between the low and high temperature limit, the so called Kramers-Wannier-Duality. Use $e^{-2K^*} = \tanh K$ to replace K^* in the low temperature limit. Calculate the critical temperature T_c by equating the free energy per spin of both temperature limits in the thermodynamic limit $N \to \infty$. Convince yourself that the critical temperature is at T_c.

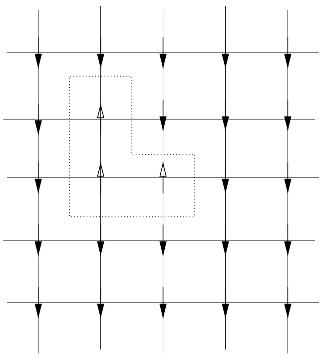

Figure 1: An example of a domain of parallel spins inside the dotted line, which are surrounded by a $n_\mathcal{D} = 8$ polygon.