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Fachbereich Physik
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Advanced Statistical Physics (WS11/12)
Problem sheet 13

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Problem 1: Boltzmann-, Bose- and Fermi-Statistics

Consider a system of $N = 2$ (spinless) particles with three one-particle-eigenstates $\epsilon_0 = 0$, $\epsilon_1 = \epsilon$ and $\epsilon_2 = 2\epsilon$. Sketch all possible configurations and calculate the canonial partition function for

a) particles, which obey the Boltzmann-statistics and are distinguishable.

b) particles, which obey the Boltzmann-statistics and are indistinguishable.

c) particles, which obey the Fermi-statistics.

d) particles, which obey the Bose-statistics.

Problem 2: Virial Coefficient for Bosons and Fermions

The effective two-particle interaction, which accounts the symmetry of the wave function, is $\tilde{v}(r) = -k_B T \ln(1 \pm e^{-2\pi r^2/\lambda^2})$, where the plus holds for bosons and the minus sign is for fermions, as shown in the lecture.

a) Sketch the potential in the form $\beta\tilde{v}(r)$ for bosons and fermions. Discuss the behavior for $r/\lambda \to 0$ and $r/\lambda \to \infty$.

b) Calculate the second virial coefficient of $\tilde{v}(r)$ by using the classical virial expansion. Determine the average particle distance a for which the contribution of the second virial term to the pressure equals the first (ideal) term? Use the particle density $c = a^{-3}$.

c) Consider three systems: A gas of electrons (i), hydrogen (ii) and oxygen (iii). Assume an average particle distance $a = 1\,\text{nm}$. At which temperature become quantum effects relevant for each system if we use the criterium of part b)?

Problem 3: Spin 1/2 Fermions in an External Magnetic Field

Consider an ideal gas of N spin 1/2 fermions confined to a volume V at zero temperature. The fermions are in an external magnetic field H. The energy of a particle is $\epsilon = \frac{p^2}{2m} \pm \mu_B H$, where μ_B is the Bohr magneton.

a) Give an expression for the chemical potential μ_0 for vanishing magnetic field as a function of the particle density N/V.

b) Calculate the average particle energy as a function of μ_0 for weak external magnetic fields.

c) Calculate the pressure $p = -\partial E/\partial V$ for vanishing magnetic field.

d) Calculate the susceptibility $\chi = \partial m/\partial H$ for weak external magnetic fields.

Problem 4: Internal Energy of Ideal Fermions at Low Temperature

We want to discuss a system of ideal (spinless) Fermions in the grand canonical ensemble for low temperature. The following expression for the internal energy has been shown in the lecture:

$$U = \frac{2\pi}{5}\frac{V\beta}{h^3m^2}\int_0^\infty p^6 \frac{e^{\beta\epsilon_p-\nu}}{[e^{\beta\epsilon_p-\nu}+1]^2}\mathrm{d}p = \frac{2^{7/2}\pi}{5}\frac{\beta V m^{3/2}}{h^3}\int_0^\infty \epsilon_p^{5/2}\frac{e^{\beta\epsilon_p-\nu}}{[e^{\beta\epsilon_p-\nu}+1]^2}\mathrm{d}\epsilon_p, \tag{1}$$

with $\nu \approx \beta\epsilon_F - \frac{\pi}{12\beta\epsilon_F}$ and the Fermi energy ϵ_F.

Since $\frac{e^{\beta\epsilon_p-\nu}}{[e^{\beta\epsilon_p-\nu}+1]^2}$ has a sharp maximum at $\beta\epsilon_p = \nu$ we can expand the factor $\epsilon_p^{5/2}$ in the integrand around $\epsilon = \nu/\beta$:

Then substitute the integration variable and use $I_0 = 1$, $I_1 = 0$ and $I_2 = \pi^2/3$ for the integral

$$I_n = \int_{-\infty}^{\infty} \mathrm{d}t \frac{t^n e^t}{(1+e^t)^2}. \tag{2}$$

Convince yourself that the expansion of the upper integrand boundary to $-\infty$ is justified.

After performing the integration insert ν and approximate up to second order terms of T to obtain the result for the internal energy as function of the Fermi energy, which has been given in the lecture.

Hint: Use $\frac{N}{V} = \frac{4\pi}{3h^3}(2m\epsilon_F)^{3/2}$, which has been shown in the lecture for spinless fermions, to obtain the final form.