Freie Universität Berlin
Fachbereich Physik
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Advanced Statistical Physics (WS11/12)
Problem sheet 2

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Problem 1: Entropy

Consider an n-sided die, where the i-th side has the probability p_i. The entropy is defined as

$$H = -\sum_{i=1}^{n} p_i \, \ln(p_i)$$

What distribution of p_i maximizes the entropy?

Problem 2: 1D Poisson Gas

Consider n randomly distributed particles in a 1-dimensional box $[0, L]$ which have the locations $x_1, ..., x_n$. The number of particles in the box is given by the Poisson distribution, where the probability that no particle is in the box is $\mathrm{e}^{-\rho L}$. The probability to find n particles in the volume corresponding to the random variable $\mathrm{d}x_1, ..., \mathrm{d}x_n$ is:

$$\frac{\mathrm{e}^{-\rho L}\, \rho^n}{n!} \mathrm{d}x_1 ... \mathrm{d}x_n, \tag{1}$$

where $\rho > 0$ is a constant and has units of inverse length.

a) Calculate the probability to find n particles in the box.

b) Show that the distribution is normalized to 1.

c) Calculate the average particle density, i.e. the average number of particles per unit length, and the variation thereof. Consider also the limiting case $L \to \infty$.

Problem 3: Moments and Cumulants

a) Express the higher moments of the deviation from the mean, $\langle \delta x^n \rangle = \langle (x - \langle x \rangle)^n \rangle$, as a function of the simple moments $\langle x^n \rangle$ (for $n = 1, 2, 3, 4$).

b) Calculate the cumulants $\langle x^n \rangle_C$ as a function of the simple moment $\langle x^n \rangle$ for $n = 1, 2, 3, 4$ using the characteristic function $G(k)$.
Hint: The expansion $\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ *is helpful.*

c) Use the normal distribution for the following exercises:

$$P(x) = \frac{\exp(-(x - x_0)^2/(2\Delta_x^2))}{\Delta_x \sqrt{2\,\pi}} \tag{2}$$

Calculate the moments $\langle x^n \rangle$ for $n = 1, 2, 3, 4$ using the characteristic function $G(k)$.

d) Calculate the higher moments of the deviation from the mean, $\langle \delta x^n \rangle = \langle (x - \langle x \rangle)^n \rangle$, for $n = 1, 2, 3, 4$.

d) Calculate the cumulants $\langle x^n \rangle_C$ for $n = 1, 2, 3, 4$ and use the definition of the cumulant by the logarithmic characteristic function $\ln G(k)$.

Problem 4: Polymer

Consider a polymer built of n rigid rods of length b. The probability distribution for the orientation of each segment in three-dimensional space is given by:

$$p(\vec{r}) = \frac{\delta(|\vec{r}| - b)}{4\,\pi\,b^2} \tag{3}$$

a) Show that the probability distribution $p(\vec{r})$ is normalized in three-dimensional space.

b) Calculate the characteristic function:

$$g(\vec{k}) = \int \mathrm{d}^3\vec{r}\, \mathrm{e}^{i\,\vec{r}\cdot\vec{k}}\, p(\vec{r}) \tag{4}$$

Solution: $g(\vec{k}) = \frac{\sin(bk)}{bk}$ *with* $k = |\vec{k}|$

c) Now look at the end-to-end distance $\vec{R} = \sum_{j=1}^{n} \vec{r}_j$. Calculate the characteristic function $G(\vec{k}) = [g(\vec{k})]^n$ and calculate the expectation values $\langle R^k \rangle$ for $k = 1, 2, 3$.

d) Calculate the expectation values of the cumulants $\langle R^k \rangle_C$ for $k = 1, 2, 3$.

Hint for (c) and (d): Approximate the (logarithmic) characteristic function by a taylor series of sufficient order.