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Advanced Statistical Physics (WS11/12)
Problem sheet 4

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Problem 1: Functional Derivative

In problem sheet 2 we have shown that the entropy of a discrete probability density is maximised in the case of a uniform distribution. Now we want to prove the same statement for a continuous probability distribution function by using functional derivatives:

The entropy of a continuous random variable with a given probability density $\omega(x)$ is defined as:

$$\mathrm{S}[\omega] := -\int_{-\infty}^{\infty} \mathrm{d}x\, \omega(x)\, \ln(\omega(x)) \tag{1}$$

Show that
a) the uniform distribution maximises the entropy among all distributions on $a < x < b$
b) for a given second moment

$$\sigma^2 = \int_{-\infty}^{\infty} \mathrm{d}x\, \omega(x)\, x^2 \tag{2}$$

and a mean of $\bar{x} = 0$ the Gaussian distribution has the maximal entropy.

Hint: Use functional derivatives and Lagrange multipliers.

Problem 2: Centrifuge

A cylinder of radius R is filled with a mixture of two gases with molecular masses m_1 and m_2. The cylinder rotates with the rotation frequency f to separate the gases. The angular velocity is $\omega = 2\pi f$. We want to discuss the long time limit, where the gas mixture is rotating with the same angular velocity as the cylinder does and follows the Boltzmann-equilibrium-distribution.

Give the force acting on the gas particles and their potential energy. Then calculate the radial density distribution function $\rho(r)$ of the two gases. Now consider a mixture of hydrogen ($m_1 = 3 \cdot 10^{-27}\,\mathrm{kg}$) and oxygen ($m_2 = 5 \cdot 10^{-26}\,\mathrm{kg}$) in a centrifuge of radius $R = 0.1\,\mathrm{m}$ at temperature $T = 300\,\mathrm{K}$.
What angular velocity ω leads to $\frac{\rho_2(R)\rho_1(0)}{\rho_1(R)\rho_2(0)} = 2$?

Problem 3: Ideal Gas in the Gravitation Field

Calculate the Helmholtz free energy F as a function of the area $A = L^2$, the temperature T and the particle number N for non-interacting particles in the external potential

$$V_{\mathrm{ext}} = \begin{cases} mgz, & \text{if } 0 \le x \le L,\ 0 \le y \le L,\ 0 \le z \\ \infty, & \text{else} \end{cases} \tag{3}$$

Calculate the density profile $\rho(z)$.

Problem 4: Ideal Dipolar Gas

We consider a non-interacting gas of N molecules with constant electric dipole moment P_0. Show that the average electric polarisation along a constant, external electric field of strength E_0 is given by the Langevin-function

$$\langle P_z \rangle = \rho P_0 \left[\coth\left(\frac{P_0 E_0}{k_B T} \right) - \frac{k_B T}{P_0 E_0} \right], \tag{4}$$

where $\rho = N/V$, volume V and temperature T are fixed.

Therefore, you should calculate:
a) at first the canonical partition function Z
b) at second the Helmholtz free energy $F(T, N, E_0)$
c) and finally, the average polarisation

$$\langle P_z \rangle = -\frac{1}{V} \left(\frac{\partial F}{\partial E_0} \right)_{T,V,N} \tag{5}$$

d) Discuss the limit of high temperatures and weak electric fields of eq. 4.