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Advanced Statistical Physics (WS11/12)
Problem sheet 5

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Problem 1: Two Level System

A system of two energy levels E_0 and E_1 is populated by N particles at temperature T. The particles populate the energy levels according to the classical distribution law.

a) Derive an expression for the average energy per particle u.

b) Compute the average energy per particle vs the temperature as $T \to 0$ and $T \to \infty$. Use reasonable approximations!

c) Derive an expression for the specific heat per particle, which is defined as

$$C = \frac{\partial u}{\partial T}.$$

d) Compute the specific heat per particle in the limits $T \to 0$ and $T \to \infty$.

Problem 2: Stationary Phase Approximation

The Gamma function is defined by the following integral representation:

$$\Gamma(x+1) = \int_0^\infty t^x \, e^{-t} \, \mathrm{d}t$$

a) Demonstrate that the gamma function exhibits the properties $\Gamma(x+1) = x\,\Gamma(x)$ and $\Gamma(1) = 1$ and therefore corresponds to the faculty for natural numbers: $\Gamma(n+1) = n!$ for $n = 1, 2, 3, \ldots$

b) Use the stationary phase approximation (*Sattelpunktsnäherung*) to derive the Stirling formula, i.e.

$$\Gamma(x+1) = \sqrt{2\,\pi\,x}\, x^x \, e^{-x}, \qquad \text{where } x \gg 1.$$

Do this by writing the integrand in exponential notation and expand the exponent around its maximum.

Problem 3: Ideal Diatomic Gas

Consider a gas consisting of classical, diatomic molecules in a volume V and at a temperature of T. The Hamiltonian is given by:

$$H(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1, \mathbf{q}_2) = (\mathbf{p}_1^2 + \mathbf{p}_2^2)/(2m) + (K/2)\,(\mathbf{q}_1 - \mathbf{q}_2)^2$$

a) Calculate the Helmholtz free energy $F(T, V)$ of that system. Assume thereby that only the center of mass of the molecules is restricted to V.

b) Calculate the entropy S, the internal energy U and the pressure $P = -\left(\frac{\partial F}{\partial V}\right)_T$ as well as the equation of state.

c) How many degrees of freedoms does this system have?

d) Calculate the expectation value for the intramolecular distance $\langle(\mathbf{q}_1 - \mathbf{q}_2)^2\rangle$.

Problem 4: One-Dimensional Chain

Consider a one-dimensional chain consisting of $n \gg 1$ segments. Let the length of each segment be a when the long dimension of the segment is parallel to the chain and zero when the segment is vertical (i.e., long dimension normal to the chain direction). Each segment has just two states, a horizontal orientation and a vertical orientation, and each of these states is not degenerate. The distance between the chain ends is nx.

a) Find the entropy of the chain as a function of x.

b) Obtain a relation between the temperature T of the chain and the tension F which is necessary to maintain the distance nx, assuming the joints turn freely.

c) Under which conditions does your answer lead to Hooke's law?

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