Freie Universität Berlin
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Statistical Physics and Thermodynamics (SS 2017)

Problem sheet 5

Hand in: Friday, May 26 during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1 Total differential (6 points)

The differential $\alpha(x,y) = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$ is called *exact* if there is a function $F(x,y)$, so that $\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}x$ is equal to α.

a) Show if α is exact, then $A(x,y)$ and $B(x,y)$ satisfy the condition

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{1}$$

(1 point)

Remark: The converse only holds for simply connected regions. In general just because equation (1) *holds this does not implies that there is a globally defined function F such that α is an exact differential* [1].

Consider $\alpha(x,y) := (cosx - xsinx + y^2)\mathrm{d}x + 2xy\mathrm{d}y$ and $\beta(x,y) := (x^2 - y)\mathrm{d}x + x\mathrm{d}y$

b) Do $\alpha(x,y)$ and $\beta(x,y)$ fullfill equation (1)? **(2 points)**

c) Are α and β exact? If yes find a function $F(x,y)$ such that $\mathrm{d}F = \alpha, \beta$ and $F(0,0) = 0$. **(2 points)**

d) Find a curve $(x, y(x))$ in space where $F(x,y) = F(\pi, 1)$ is constant. Plot $y(x)$ in the range $-2\pi < x < 2\,\pi$. **(1 point)**

2 Entropy of two subsystems (7 points)

Consider two subsystems with the number of microstates $\Gamma_1(U_1)$ and $\Gamma_2(U_2)$ and their energies U_1 and U_2 respectively. The subsystems can exchange energy so that the total energy of the system is $U = U_1 + U_2$.

a) Write down the number of microstates of the total system $\Gamma(U)$. Use that $\ln\Gamma(U) = S(U)/k_B$ to express the total entropy $S(U)$ in terms of the entropies of the two subsystems $S_1(U_1)$ and $S_2(U_2)$. **(1 point)**

b) Calculate $S(U)$ for the explicitly given entropies $S_1(U_1) = -a_1(U_1 - U_1^0)^2$ and $S_2(U_2) = -a_2(U_2 - U_2^0)^2$ with $a_1 > 0, a_2 > 0$. **(3 points)**

c) The entropy and the energy are extensive quantities, so they scale with the number of particles N. What does this imply for the scaling of a_1 and a_2? **(1 point)**

d) Using your results from c), how do the terms in $S(U)$ scale? Is your result extensive for large N. **(1 point)**

e) What is the temperature T of both systems and how does this scale with the number of particles N? **(1 point)**

[1] https://en.wikipedia.org/wiki/Closed_and_exact_differential_forms

3 Scale heights (7 points)

In the following you will derive the "scale height", an estimate for the atmospheric height of a planet. Assume that all particles in the atmosphere got the mass m and are in thermal equilibrium at temperature T regardless of the their height h above the planet's surface. The particles are subjected to a gravitational potential with the energy $U = -GmM/(R+h)$, with G the gravitation constant, M the planet's mass and R its radius.

a) Expand U for small h up to the linear order and write down the non normalized probability distribution $\rho(h)$ for the approximated potential. Use $g = GM/R^2$ as an abbreviation for the gravitational acceleration. Argue up to which height your approximation is valid. **(2 points)**

Hint: For the validity of your expansion consider higher order terms.

b) Normalize the distribution in terms of k_B, T, m and g for the approximated potential. Is the exact distribution normalizable? If not, what does this mean physically? **(2 points)**

c) Calculate the mean (scale height) and its standard deviation **(2 points)**

The following table lists some characteristics of Earth and Mars

	Earth	Mars
radius (R)	6371 km	3390 km
gravitational acceleration (g)	9.82 m/s^2	3.74 m/s^2
average temperature (T)	287 K	226 K
mass of atmospheres particles (m)	28 u (N_2)	$44,01$ u (CO_2)

d) Calculate the atmospheric scale heights and its standard deviation for Earth and Mars ($k_B = 1.38065 \times 10^{-23} J/K$). Is your approximation of a) still valid? **(1 point)**