Freie Universität Berlin
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May 22nd 2018

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Statistical Physics and Thermodynamics (SS 2018)

Problem sheet 5

Hand in: Friday, June 1st during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1 Maxwell-Boltzmann distribution in 2 dimensions (8 points)

Consider a particle of mass m moving in a 2-dimensional plane with isotropic velocity (v_x, v_y). The Hamiltonian in terms of the speed $v = \sqrt{v_x^2 + v_y^2}$, defined as the magnitude of the velocity vector, is given by $\mathcal{H} = mv^2/2$.

a) Write down an expression for the second moment of the speed, $\langle v^2 \rangle$, in terms of an integral over v. **(2 points)**

b) Using that the second moment can be expressed as

$$\langle v^2 \rangle = \int_0^\infty dv v^2 \rho_{\mathrm{MB}}(v),$$

derive the expression for the normalized Maxwell-Boltzmann distribution of the speed, $\rho_{\mathrm{MB}}(v)$. **(2 points)**

c) Calculate the average speed $\langle v \rangle$ and the most likely speed, given by the maximum of $\rho_{\mathrm{MB}}(v)$. **(2 points)**

d) Calculate the second and the third cumulants of $\rho_{\mathrm{MB}}(v)$, given by

$$\langle v^2 \rangle_C = \langle v^2 \rangle - \langle v \rangle^2 \qquad \text{and}$$

$$\langle v^3 \rangle_C = \langle v^3 \rangle - 3\langle v^2 \rangle \langle v \rangle + 2\langle v \rangle^3,$$

respectively. What does it mean that the third cumulant is nonzero? **(2 points)**

2 Equipartition theorem for non-quadratic Hamiltonians (7 points)

Consider a system with a Hamiltonian $\mathcal{H} = A|q|^m$, with A being a positive number.

a) Write down the partition function Z for a single positional degree of freedom q. **(2 points)**

b) Calculate the energy U from a derivative of $\ln Z$ using a suitable coordinate transform. **(3 points)**

c) Calculate the heat capacity of the system $C = (\partial U / \partial T)$. **(2 points)**

3 Energy of an oscillator (5 points)

Consider a one-dimensional harmonic oscillator with Hamiltonian $\mathcal{H} = p^2/(2m) + kq^2/2$, with p being the momentum, q being the position, m being the mass and k being a spring constant.

a) Write down the partition function Z and calculate the internal energy U of the oscillator from a derivative of $\ln Z$. **(1 point)**

b) Do the same as in part a) for (i) an anharmonic oscillator with Hamiltonian $\mathcal{H} = p^2/(2m) + kq^4/2$ and (ii) an anharmonic oscillator with Hamiltonian $\mathcal{H} = p^2/(2m) + k|q|/2$. Compare your results to the result of part a). **(4 points)**

Hint: The integral $\int_{-\infty}^{\infty} \exp\left[-Aq^4/2\right] dq = 2^{-\frac{3}{4}}\,\Gamma(\frac{1}{4})\,A^{-\frac{1}{4}}$, *with* $\Gamma(x)$ *being the Gamma function.*