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21(23).4.2015

Advanced Statistical Physics II – Problem Sheet 0

Problem 1 – **Gaussian Integrals**

a) Calculate

$$I \equiv \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \tag{1}$$

by expressing I^2 in polar coordinates.

b) Now obtain a result for

$$\int_{-\infty}^{\infty} dx\, e^{-a\frac{x^2}{2}}. \tag{2}$$

c) Aditionally, evaluate the following integrals:

$$\int_{-\infty}^{\infty} dx\, xe^{-\frac{x^2}{2}}, \tag{3}$$

$$\int_{-\infty}^{\infty} dx\, x^2e^{-\frac{x^2}{2}}. \tag{4}$$

d) A further generalization are integrals of the form

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}-bx}. \tag{5}$$

Find a solution by using your above results.

e) Now we want to consider the more interesting case of n-dimensional Gaussian integrals with arbitrary positive definie symmetric bilinear forms, i.e. consider a symmetric matrix $G = (g_{ij})$ with positive eigenvalues (λ_i) and calculate the integral

$$\int d^n x\, e^{-\frac{1}{2}\vec{x}^T G\vec{x}} = \int d^n x\, e^{-\frac{1}{2}x_i g_{ij} x_j}. \tag{6}$$

Hint: Remember the spectral theorem.

f) Finally, solve the integral

$$\int d^n x\, e^{-\frac{1}{2}\vec{x}^T G\vec{x}-\vec{b}\cdot\vec{x}}. \tag{7}$$

Problem 2 – **Euler Theorem**

The Euler theorem for homogeneous functions says that if a differentiable function $f(x_1, ..., x_k)$ is homogeneous of degree m, i.e. if

$$f(\lambda x_1, ..., \lambda x_k) = \lambda^m f(x_1, ..., x_k), \tag{8}$$

then it follows that

$$\frac{\partial f}{\partial x_1}x_1 + ... + \frac{\partial f}{\partial x_k}x_k = mf. \tag{9}$$

(This can easily be proven by taking the derivative with respect to λ on both sides of eq. 8 and then setting $\lambda = 1$.)

a) Use the Euler homogeneous function theorem to derive the following version of the internal energy

$$U = TS - PV + \mu N \tag{10}$$

b) Similar show that

$$\Omega = -PV \tag{11}$$

c) From (b) you can derive why the pressure P in the grand potential does not deppend on the volume V or the particle number N:

$$P = P(\mu, T). \tag{12}$$

Problem 3 – **Fourier Transform**

a) Consider a vibrating long string whose amplitude $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \tag{13}$$

i) Defining $Y(k,t)$ via the Fourier transform:

$$Y(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,t)e^{ikx}dx, \tag{14}$$

convert the partial differential equation (PDE) Eq. (13) into an ordinary differential equation (ODE) of Y.

ii) Using the initial condition $y(x, t = 0) = y_0(x)$ and its Fourier transform $Y_0(k)$, solve the ODE for $Y(k,t)$.

iii) Perform the inverse-Fourier transfrom of $Y(k,t)$ and show

$$y(x,t) = y_0(x \mp vt). \tag{15}$$