Freie Universität Berlin
Fachbereich Physik
Prof. Dr. Roland Netz
http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre

Hand in on May 15th 2015 before the lecture

Advanced Statistical Physics II – Problem Sheet 4

Problem 1 – **Fluctuations of a single particle in quadratic potential**

Consider a small particle that is, for example via an optical tweezer, confined to some region of space via a potential

$$V(\vec{x}) = \frac{1}{2}\vec{x}^T K \vec{x}, \tag{1}$$

where $\vec{x} \in \mathbb{R}^3$ is the position of the particle and K is a positive definite symmetric 3x3 matrix [positive definite : $\vec{x}^T K \vec{x} > 0$, for non-zero vector $\vec{x}$]. Due to Brownian motion (which will be discussed in more detail later in the lecture), the particle does not stay at the equilibrium position $x = 0$ all the time but fluctuates around, resulting in a probability distribution for its position that is given by

$$\mathcal{P}(\vec{x}) = C \exp\left[-\frac{1}{2k_B T}\,\vec{x}^T K \vec{x}\right], \tag{2}$$

with C the normalization constant. By $F_i(\vec{x})$ we denote the force in i-direction the potential excerts on the particle if it is at position $\vec{x}$.

a) Calculate C.

b) Find the values of $\langle F_i(\vec{x})\rangle$ and $F_i(\langle\vec{x}\rangle)$.

c) Calculate the correlations $\langle x_i x_j\rangle$, $\langle x_i F_j(\vec{x})\rangle$ and $\langle F_i(\vec{x}) F_j(\vec{x})\rangle$.

d) Consider a potential

$$V(\vec{x}) = \frac{1}{2}\vec{x}^T K \vec{x} + \vec{b}\cdot\vec{x}, \tag{3}$$

where $K = \left(\begin{smallmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{smallmatrix}\right)$ with $k > 0$, and $\vec{b} = (b, b, b)$. Calculate the correlations $\langle x_i x_j\rangle$, $\langle x_i F_j(\vec{x})\rangle$ and $\langle F_i(\vec{x}) F_j(\vec{x})\rangle$. Compare the correlations for $b = 0$ with the expressions found in c).

Problem 2 – **Particles trapped in a cone-like potential**

Consider N non-interacting distinguishable particles subject to a cone-like potential $V(\rho_\alpha) = U_0\rho_\alpha$ per particle with the index $\alpha = 1, \ldots, N$, in two-dimensional polar coordinates $\vec{r}_\alpha = (\rho_\alpha\cos(\theta_\alpha), \rho_\alpha\sin(\theta_\alpha))$, where ρ_α is the radial coordinate, and θ_α is the angular coordinate.

a) Calculate the partition function

$$Z_N = \left(\frac{2\pi}{h^2}\right)^N \prod_{\alpha=1}^{N} \int_0^{2\pi} d\theta_\alpha \int_0^\infty d\rho_\alpha \rho_\alpha \int_0^\infty dp_\alpha p_\alpha \exp\left[-\frac{\beta}{2m}p_\alpha^2 - \beta U_0 \rho_\alpha\right], \tag{4}$$

where h is the Planck constant, and p_α is the momentum.

b) In terms of $k_B T$ and U_0, find the average position $\langle\vec{r}_\lambda\rangle$, average radial distance $\langle\rho_\lambda\rangle$, and position fluctuation $\langle\rho_\lambda^2\rangle$, where $\lambda \le N$ is a certain positive integer.

c) Find the average energy $\langle U\rangle$.

d) Find the occupation number ratio $n(\rho)/n(\rho = 0)$, where $n(\rho)$ is the average number of particles at the radial distance ρ. [Hint: For this consider the probability to find a particle at the radial distance ρ.]

Problem 3 – **Fluctuations at fixed volume**

Consider a system where the particle number N and energy U can fluctuate in a fixed volume V.

a) Find a matrix g_{ij} where $i = U, N$ and $j = U, N$, and express $\langle \Delta N \Delta N \rangle$, $\langle \Delta N \Delta U \rangle$ and $\langle \Delta U \Delta U \rangle$ in terms of k_B, N, κ_T, T, V and C_V.

b) Using the above results and $\langle f_n f_m \rangle = k_B g_{nm}$ as well as $\langle a_n f_m \rangle = -k_B \delta_{nm}$ which was given in the lecture, express $\langle \Delta N \Delta(T^{-1}) \rangle$ and $\langle \Delta(T^{-1})(\Delta T^{-1}) \rangle$ in terms of k_B, N, κ_T, T_0, V and C_V. Note that $\Delta(T^{-1}) = 1/T - 1/T_0$, and T_0 is the temperature of the reservoir.

c) Find $\langle \Delta T \Delta T \rangle$ for $\Delta T \ll T_0$ and compare with $\langle \Delta(T^{-1})(\Delta T^{-1}) \rangle$.