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hand in on 22th May 2015 before lecture

Advanced Statistical Physics II – Problem Sheet 5

Problem 1 – **Conservation of Angular Momentum** *Note that all quantities in eq (1) depend on time and space.*

Starting from the balance equation for linear momentum,

$$\frac{\partial \rho v_i}{\partial t} + \nabla_j (v_i \rho v_j - \sigma_{ij}) = \rho F_i, \tag{1}$$

we will derive the balance equation for angular momentum. From this equation we will then derive that angular momentum conservation implies the stress tensor σ_{ij} to be symmetric.

Before we begin, recall that cross products can be written as

$$(\vec{v} \times \vec{w})_i = \epsilon_{ijk} v_j w_k, \tag{2}$$

where the Levi-Civita symbol ϵ_{ijk} is given by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{if } (i,j,k) \in \{(1,3,2),(3,2,1),(2,1,3)\} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

a) Analogously to point particles, we define in the continuum the angular momentum density

$$\vec{L} = \vec{r} \times \rho \vec{v} \tag{4}$$

and the torque density

$$\vec{M} = \vec{r} \times \rho \vec{F}. \tag{5}$$

Take the cross product of the vector equation (1) with $\vec{r}$ and write the result in the form of a balance equation for the angular momentum density,

$$\frac{\partial L_i}{\partial t} + \nabla_j J^L_{ij} = Q^L_i, \tag{6}$$

thereby defining the angular momentum flow density tensor $\overleftrightarrow{J}^L$ and the angular momentum production $\vec{Q}^L$.

b) Assuming the torque density vanishes and angular momentum is conserved, use equation (6) to show that the stress tensor is symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$.

Problem 2 – **Mass Conservation**

Consider a system of K components described by the continuous mass densities $\rho^{(i)}(\vec{r},t)$, with $i = 1,\ldots,k$. Recall that by definition of the $\rho^{(i)}$, the mass of component i within any fixed volume V is given by

$$m^{(i)}(V) = \int_V \rho^{(i)} dV. \tag{7}$$

We assume that no chemical reactions take place, so that there is no interconversion between different components.

a) Conservation of mass for the i-th component can be formulated as the statement that for every volume V, the change of mass within V is equal to the net flow through the volume's surface S, i.e.

$$\frac{d}{dt}m^{(i)}(V) = -\int_{S(V)} \rho^{(i)} v_j^{(i)} dS_j. \tag{8}$$

Here, $v_j^{(i)}$ denotes the j-component of the velocity vector $\vec{v}^{(i)}$, which describes the flow of the i-th component of the system. Note that in Eq. (8) there is no sum over i but a sum over j.

Use the Gauss law and the fact that eq. 8 holds for arbitrary volume to formulate the conservation of mass equation in differential form, i.e. without integrals.

b) Now sum over i and write the resulting conservation law for the total mass in terms of the total mass density ρ and the center-of-mass velocity

$$v_j = \frac{\sum_i \rho^{(i)} v_j^{(i)}}{\rho}. \tag{9}$$

c) Reformulate the conservation law for the total mass using the *substantial time derivative* (also called *material derivative* or *convective derivative*)

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_j \nabla_j, \tag{10}$$

i.e. show that

$$\frac{d\rho}{dt} + \rho \nabla_j v_j = 0. \tag{11}$$

d) Show that for a total mass density that propagates with constant velocity, $\rho(\vec{r}, t) = f(\vec{r} - \vec{v}t)$ (where $\vec{v}$ is a constant) the convective derivative is zero.

e) In the lecture the continuity equation in a fixed frame was given as

$$\frac{\partial \rho(\vec{r}, t)}{\partial t} = -\vec{\nabla} \cdot [\rho(\vec{r}, t)\vec{v}(\vec{r}, t)]. \tag{12}$$

Consider a one-component particle system in a fixed volume V, in three-dimensional spherical coordinates $\vec{r} = (r, \theta, \phi)$ where $0 \le r$ is the radial coordinate, $0 \le \theta \le \pi$ is the angle with the z axis (polar angle), and $0 \le \phi \le 2\pi$ is the angle rotating aroud the z axis (azimuthal angle).

i) Consider a velocity field $\vec{v} = (v_r = 0, v_\theta = 0, v_\phi(r))$. Find the continuity equation for the total mass density $\rho(\vec{r}, t)$.

Hint: Use the gradient and the divergence in the spherical coordinates.

ii) Consider a total mass density that is uniform in space and changing in time, $\rho(t) = \rho_0 \cos^2(\Omega t) e^{-t/T}$ with $\{\rho_0, \Omega, T\} > 0$. Find the radial velocity field $v_r(r, t)$ with the boundary condition $v_r(r = 0, t) = 0$ and $v_\theta = v_\phi = 0$ for all times.

Problem 3 – **Stress Tensor**

Consider a resting continuum body B subject to a body force F_i and a surface stress $p_i = \sigma_{ij} n_j$ on its bounding surface. Suppose B is in equilibrium and let σ_{ij} denote its stress tensor. Define the average stress tensor $\overline{\sigma_{ij}}$ (which is a constant) in B by

$$\overline{\sigma_{ij}} = \frac{1}{V} \int_V \sigma_{ij} dV \tag{13}$$

a) Use the local equilibrium condition for zero velocity from the lecture

$$\nabla_j \sigma_{ij} + F_i = 0 \tag{14}$$

to show that

$$\overline{\sigma_{ij}} = \frac{1}{V} \int_V x_i F_j dV + \int_{S(B)} x_i p_j dA. \tag{15}$$

Hint: Note that the stress tensor can be written as

$$\sigma_{ij} = \nabla_k (x_j \sigma_{ik}) - x_j \nabla_k \sigma_{ik} \tag{16}$$

Remark: The above expression is known as the Signorini's theorem, which states that the average value of the stress tensor in a body in equilibrium is completely determined by the external surface stress p_i and the body force F_i

b) Suppose that $F_i = 0$ and the applied surface stress p_i is a uniform pressure

$$p_i = -p_0 n_i, \tag{17}$$

where p_0 is a constant and n_i is the outward unit normal field on the surface of B. Use the result in a) to show that the average stress tensor is

$$\overline{\sigma}_{ij} = -p_0 \delta_{ij}. \tag{18}$$

c) Show that in b) the pressure $-p$ is given by the arithmetic average of the eigenvalues (principle stresses) of the average stress tensor.

d) (**Archimedes' principle**) Now consider B is immersed in a liquid of constant mass density ρ and is subject to a uniform gravitational force field per unit mass. In this case, the liquid exerts a hydrostatic surface force field $p_i = -p_0 n_i$ on the bounding surface of B, where $p = \rho g x_3$ is the hydrostatic pressure in the liquid, and g is the gravitational acceleration constant. Show that the hydrostatic surface force on B (the *buoyant force*) is given by

$$\mathbf{F}^{(buoyant)}\big|_{S(B)} = -gM\mathbf{e}_3, \tag{19}$$

where M is the mass of the liquid of the geometrical form of B.

e) Suppose the stress tensor at a point $\mathbf{x}$ in a body has the form

$$\overleftrightarrow{\sigma} = \begin{pmatrix} 5 & 3 & -3 \\ 3 & 0 & 2 \\ -3 & 2 & 0 \end{pmatrix} \tag{20}$$

and consider a surface S with surface normal $\mathbf{n} = (0, 1/\sqrt{2}, 1/\sqrt{2})^T$ and another surface S' with normal $\mathbf{n}' = (1, 0, 0)^T$ at $\mathbf{x}$.

i) Find the normal and tangential surface stresses $\mathbf{p}_\perp$ and $\mathbf{p}_\parallel$ on S and S'. In particular, show that S experiences no shear traction, whereas S' does.

ii) Find the principal stresses and stress directions of σ_{ij} and verify that $\mathbf{n}$ is a principal direction.
Remark: Principal stresses are stresses, that have only a normal-stress component and no tangential-stress component. Their respective directions are eigenvectors of σ_{ij}.

$$\mathbf{p} = \mathbf{p}_\perp + \mathbf{p}_\parallel = \mathbf{p}_\perp.$$